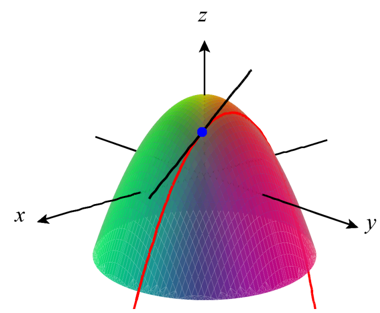
Partial Derivatives Activity Instructor Notes:

Although students have years of experience graphing in the 2D-plane, not all of their intuition and skills transfer to the spatial thinking needed in 3D. For example, when interpreting the slope of a tangent line for a function of one variable, students naturally read graphs from left-to-right in the 2D-plane. This left-to-right thinking does not apply to interpretations of partial derivatives for functions of two variables, as shown in the figure here, where students may often misinterpret f_x to be positive at the given point. To help students practice the spatial concepts of partial derivatives, we created an activity using 3D models in which they identify points with specific derivative properties on the 3D surface.



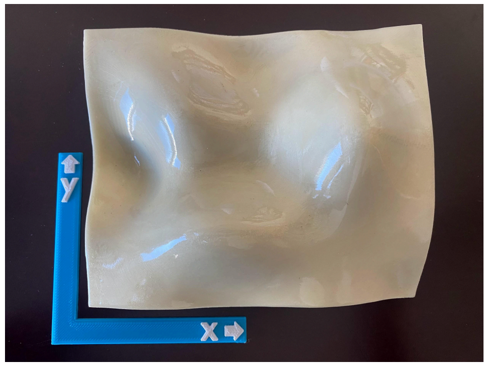
Instructors may choose different times at which to implement this activity during their lessons on partial derivatives. It could work well toward the beginning, middle, or end of the topic coverage. Before students embark on the activity, it is important that they have had some kind of geometric conceptual introduction to partial derivatives. In Paul's class, he uses built-in features in CalcPlot3D to discuss first and second partial derivatives as in this example. Here, you can click the point in the 2D plane on the left and drag it around (or use the sliders) to view the corresponding tangent line whose slope illustrates the partial derivative with respect to *x* on the surface at the specified point. In Shelby's class, partial derivative concepts are introduced through a field trip to a hill on campus inspired by Eric Miles' MAA Focus Article. The hill activity takes about 25 minutes, and the remaining 25 minutes of class is used for this Partial Derivatives Activity with the surfaces.

Through the activity and guided worksheet, students will:
- Extend the concepts of slope and concavity to the 3D setting of functions of two variables
- Practice geometric concepts of first and second partial derivatives
- Foreshadow critical points, max/min/saddle points
- Interpret partial derivatives from contour plots

Materials:
- Scaffolded worksheet (Modify it however you wish for your class! We would love to hear about how you implement the worksheet, so please feel free to reach out to Paul and Shelby to share.)
- 3D models (3D-printed or thermoformed)
  - We suggest using the project mountainous surfaces (STL files can be accessed here) or you can choose to design your own surfaces.
  - Thermoformed models are fine for this activity, but 3D-printed models are slightly easier for students to use because the transparency of the thermoformed models makes it more challenging to perceive the slopes.
  - We recommend one surface per 2-3 students.
- Whiteboard marker or wet erase marker to write on the model
  - 3D-printed model sealed with XTC 3D Smoother: whiteboard marker
  - 3D-printed model with no sealant: wet erase marker works well, but have wet cloths on hand to erase if you teach back to back sections.
  - Thermoformed model: whiteboard marker

Developed by Shelby Stanhope and Paul Seeburger for Taking CalcPlot3D to the Next Dimension, NSF – IUSE #2121152 and #2121133. These materials can be accessed at: https://sites.monroecc.edu/multivariablecalculus/3d-learning-activities/

- Some method to denote the *x*- and *y*-directions. This could be done simply using a piece of paper underneath the model, or with 3D-printed, labeled axes as shown in this picture.



Implementation Notes:

- We walk around during the activity asking students questions to assess their understanding and to assist groups that may be struggling on various parts of the worksheet.
- You can choose how you wish to grade this assignment.
  - Paul collects the marked surfaces (labeled with a letter for each group) and grades this activity, giving bonus points for correctly locating a point for #5 and #8.
  - Shelby does not assign points to this in-class activity and does not grade their work.

Partial Derivatives Activity (With Instructor Notes)

Activity Setup: Your group has a surface that is the graph of a function of two variables we'll call f.

Mark the sides of your surface to show your choice for the direction of the positive x- and positive y-axes. Be sure your choice will result in the positive z-axis pointing upward (in a right-hand coordinate system).

Activity Tasks: Locate points with the following properties on your 3D-printed surface. Use the marker to draw a point there and label it with the corresponding number below.

After finding the location of a point with the specified properties, write a "non-mathy" description of what the surface is doing at the corresponding point. (For example: downhill in the positive x-direction or as another example, sloping down in the positive x-direction).

1. a point where f_x is a small positive number

   ____
2. a point where f_y is a large negative number

   ____
3. a point where $f_x > 0$ and $f_y < 0$

   ____
4. a point where $f_x > 0$ and $f_y > 0$

   ____
5. a point where $f_x = 0$ and $f_y \neq 0$

   (For my students, this is typically the first problem they struggle with)______________________
6. a point where $f_x = 0$ and $f_y = 0$

   ____

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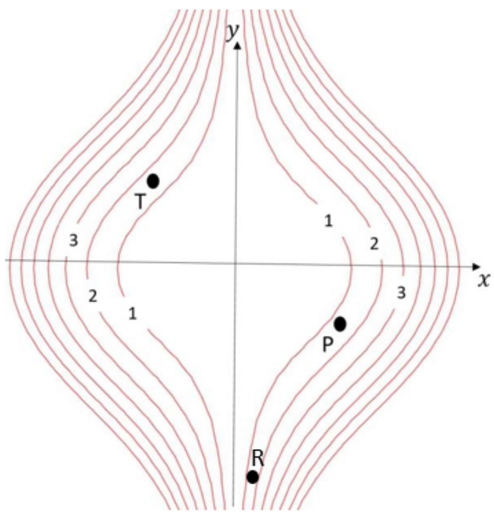
7. a point where $f_{xx} > 0$ and $f_{yy} > 0$ or where $f_{xx} < 0$ and $f_{yy} < 0$ (Circle which you used)

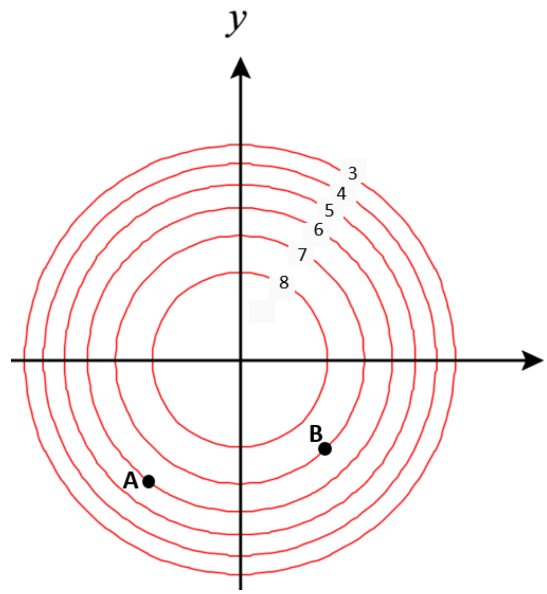
____

8. a point where $f_{xx} > 0$ and $f_{yy} < 0$

_(This question also tends to be more challenging for students)____________________

Contour Plot Activity: Below are the contour plots for two functions of two variables, with contour elevations labeled.





1. At each of the labeled points, determine if f_x and f_y are positive, negative, or zero.

(These questions add another layer of thinking to the tasks they just practiced, because they need to understand what the contour plot tells us about the behavior of the surface near a given point.)

2. What is the sign of $f_{xx}(P)$? Explain why you think this is likely.

(This problem assesses their understanding of using contour spacing to discuss second partial derivatives. At the point *P*, $f_x(P)$ is positive, but as you move in the *x*-direction, the contours are closer together, indicating steeper positive slopes, so f_x is increasing in the *x*-direction, thus $f_{xx}(P)$ is positive)

3. In the graph on the left, which quantity is larger $f_x(P)$ *or* $f_x(R)$? Why?

(This circles back to concepts about contour spacing. Both partial derivatives are positive, but it is larger at point R because the surface is steeper there.)

4. In the graph on the right, mark a point *M* (not at the origin) such that $f_y(M) = 0$.

(The most obvious point is at the max at the origin, but we want students to be able to find a non-critical point where the y partial is zero (anywhere on the x-axis).)

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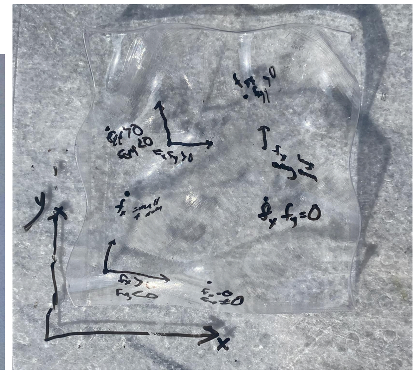
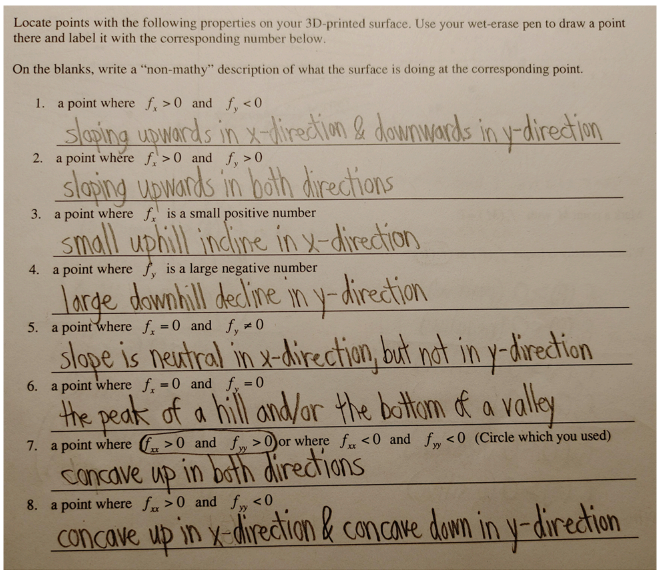
Here are a few pictures of students and their work:





Locate points with the following properties on your 3D-printed surface. Use your wet-erase pen to draw a point there and label it with the corresponding number below.

On the blanks, write a "non-mathy" description of what the surface is doing at the corresponding point.

1. a point where $f_x > 0$ and $f_y < 0$
   sloping upwards in x-direction & downwards in y-direction
2. a point where $f_x > 0$ and $f_y > 0$
   sloping upwards in both directions
3. a point where f_x is a small positive number
   small uphill incline in x-direction
4. a point where f_y is a large negative number
   large downhill decline in y-direction
5. a point where $f_x = 0$ and $f_y \neq 0$
   slope is neutral in x-direction, but not in y-direction
6. a point where $f_x = 0$ and $f_y = 0$
   the peak of a hill and/or the bottom of a valley
7. a point where ($f_{xx} > 0$ and $f_{yy} > 0$) or where $f_{xx} < 0$ and $f_{yy} < 0$ (Circle which you used)
   concave up in both directions
8. a point where $f_{xx} > 0$ and $f_{yy} < 0$
   concave up in x-direction & concave down in y-direction

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