Constrained Optimization - Lagrange Multipliers Activity with Script Instructor Notes:

Constrained optimization is typically covered after the Second Derivatives Test for local extrema. In constrained optimization, we are asking a fundamentally different question. Instead of looking for the peaks and valleys of a function of two variables, we are looking for the high and low points on a constrained part of the surface (typically a closed curve on the surface). The method of Lagrange multipliers to solve constrained optimization problems is often viewed by students as a black-box exercise (a.k.a "math magic") that requires a bunch of algebra. This activity aims to help students understand the graphical concepts of constrained optimization and steps through a geometric derivation of the Lagrange equations, allowing us to revisit several important multivariable concepts along the way.

Through the activity and guided worksheet, students will:

- Graphically visualize a function of two variables subject to a constraint.
- Review the relationship between the level curves and gradient vectors.
- Geometrically derive the Lagrange equations through parallel gradient vectors.
- Summarize the method of Lagrange multipliers.

Materials:

- Scaffolded worksheet (Modify it however you wish for your class! We would love to hear about how you implement the worksheet, so please feel free to reach out to Paul and Shelby to share.) Bringing printed copies of the worksheet to class (especially page 2) is important.
- 3D models (thermoformed or otherwise transparent)
    - We suggest using the project mountainous surfaces (STL files can be accessed here) or you can choose to design your own surfaces.
    - We recommend one surface per 2-3 students.
- Whiteboard marker to write on the model
- CalcPlot3D (via laptop, tablet, or phone). In general, the experience will be better on a laptop or tablet.

Implementation Notes:

- In this guide we describe how Shelby has used this activity in class, making heavy use of instructor guidance to facilitate a class discussion and a CalcPlot3D script. Paul uses a more inquiry based approach for this activity, and his instructor notes are provided here.

Constrained Optimization - Lagrange Multipliers Activity with Script (With Notes)

(Students in groups of 2-3 have a clear thermoformed mountainous surface, a whiteboard marker, and this scaffolded worksheet, including the contour plot on page 2.)

Consider the mountainous terrain of the clear plastic model, and let $z = f(x, y)$ represent the elevation at any (x, y) location. Suppose you are taking a hike along a circular route, $g(x, y) = c$ (we'll call this our constraint).

To visualize your hike, place the contour plot underneath your clear plastic surface so that the upper-left and lower-left corners of the contour plot match these edges of the top of the surface and the contour plot matches the surface above. Use the dry-erase marker to trace the trail you would walk on this circular path along the mountain on the surface. (Helpful hint: It is easiest to stand above the surface, close one eye, and look straight down to draw the circular path).

(Most students are able to draw the path onto the surface easily.)

Observe the path you drew through this mountainous terrain. On the surface, mark a large dot at all of the relative high points and low points you encountered along your hike. Then indicate with an "x" the absolute highest and lowest elevations that you encountered during your hike.

Developed by Shelby Stanhope and Paul Seeburger for Taking CalcPlot3D to the Next Dimension, NSF – IUSE #2121152 and #2121133. These materials can be accessed at: https://sites.monroecc.edu/multivariablecalculus/3d-learning-activities/

Let's observe a key relationship between the constraint curve and the contours at these points. [Calcplot3D Link](step 3)

(This script corresponds to the white mountain, but students who happen to have the green or yellow mountain models don't typically have a problem transitioning to the new setting.

I have my computer screen, with the CalcPlot3D script open, projected onto a whiteboard for this activity. By doing this, I can invite student volunteers to draw items on the board that have relevance to the projected graphs.

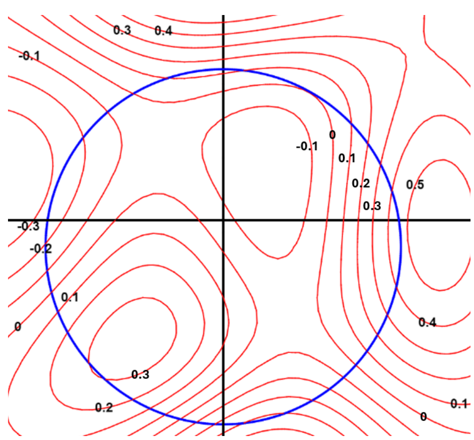
Step 1 of the script: shows in CalcPlot3D, what they have just drawn on their surface. In fact, step 1 can replace the use of a physical model for classrooms that do not have access to clear models.

Before moving on to Step 2, I invite a student up to the board to draw on the projected graph, the low and high points encountered on the hike. If you cannot project onto a whiteboard, you can invite someone to point them out on the screen.

Step 2 adds these low and high points to the graph (it is usually fun for the students to see how close they got with the points they drew). Next, I aim to motivate the method of Lagrange multipliers. The text of the script captures the essence of the conversation I have with the students "These are the 6 high and low points you encountered along the way. From among these, we can quickly identify the absolute max and absolute min elevations we reached on our hike. Notice that there are infinitely many locations that we encountered along our hike. So finding where the absolute max/min are located sounds like a very challenging endeavor! However, these 6 points here are the ONLY points we need to consider when searching for the absolute max/min. Through the remaining discussion here, we will develop a method to identify this finite set of points of interest!"

Step 3: In this step, the contours are added to the 3D surface, and we observe a key relationship between the constraint curve and the contours at these points. I am waiting for students to chime in with the observation that they are "tangent" or "parallel" at these points. I also like to explain that as I am walking up the hill, I am crossing contours (because my elevation is changing) but the moment I arrive at a high point, I stop crossing contours, and I am tangent to a level curve right before I start to head downhill. This is also a great opportunity to discuss how a contour map only graphs a sample of level curves. (What would happen if we graphed all of them?) So, I often draw in a missing level curve that would pass through one of the high points.

Next, on the 2D contour plot below, mark the points you found that were high points and low points on the surface (using the same symbols). Remember to stay on your circular hiking trail. (see step 4 in the CalcPlot3D Link)



Step 4: In this step, we take the conversation into 2D because ultimately we need to talk about gradients, and gradients of functions of two variables are vectors in 2D. Keeping in mind that the level curves and the constraint are tangent/parallel at the points of interest, we draw in those points onto this 2D contour plot. Again, I invite a student to the board to draw them in.

Developed by Shelby Stanhope and Paul Seeburger for Taking CalcPlot3D to the Next Dimension, NSF – IUSE #2121152 and #2121133. These materials can be accessed at: https://sites.monroecc.edu/multivariablecalculus/3d-learning-activities/

Follow steps 5-9 to derive the Lagrange Equations:

Step 5 adds these points to the 2D graph (again, the students enjoy seeing how close the volunteer got to the points).

Step 6 has the same 2D graph, but with the constraint removed, so that we can focus on the contour plot of the mountains and the points of interest. The script text prompts students to pick two of these points and to draw a vector in the direction of the gradient of f at these two points. The reason I do not ask them to draw ∇f at the point is because I don't want them to have to worry about making the magnitude of the vector accurate. It is easier to just draw a vector in the direction of ∇f using the fact that it needs to be perpendicular to the level curve and going uphill. For my students, this exercise is review. When we covered gradients a few weeks prior, we practiced drawing a vector in the direction of the gradient at a point. Here, a student volunteer comes to the board to draw a vector in the direction of the gradient of f at two points. (They usually choose the 2 points that pass through a level curve, in upper right and lower left).

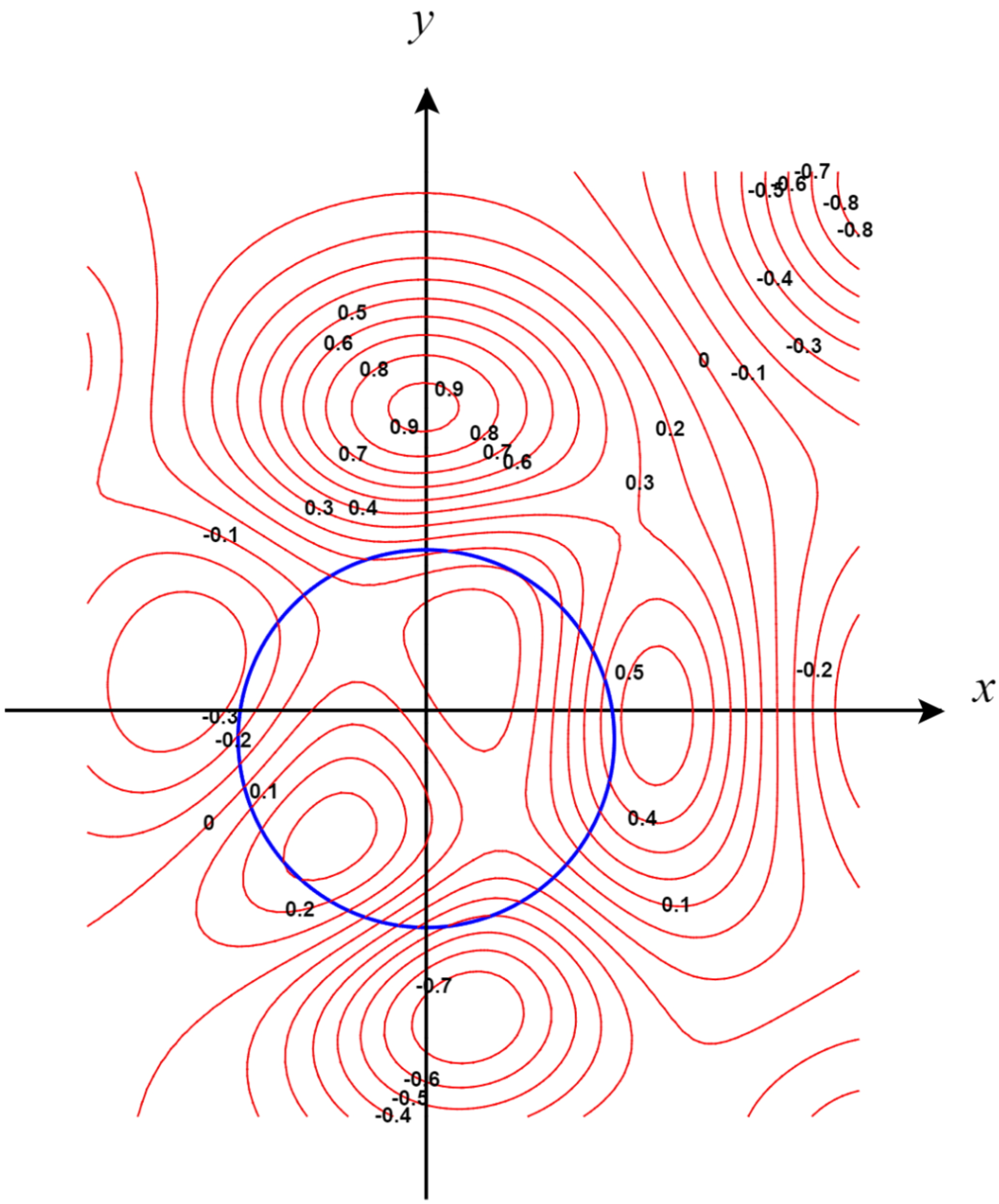
Step 7: In this step, now we only view the constraint curve in 2D along with the points of interest. Now, we need to establish that this constraint is actually just one level curve of a function $z = g(x, y)$. This is also a nice opportunity to review how we compute level curves by setting the function equal to a constant, $g(x, y) = c$. The script text reads "Our path (constraint) is a circle here, which is some $g(x, y) = c$ curve (for example, $x^2 + y^2 = 1$). So, we can view the constraint $g(x, y) = c$ as one level curve of a function $z = g(x, y)$."
This is typically the most challenging discussion point of the activity. If students can buy in that this is a level curve of g, then asking them to draw a vector in the direction of ∇g just requires them to draw a vector perpendicular to that curve at the point. Which way does it point? I try to not harp on this too much. (In fact we really don't know which way the gradient points because we only have 1 level curve. For example, for the constraint $x^2 + y^2 = 1$, this is a level curve for both $g(x, y) = x^2 + y^2$ and for $g(x, y) = 2 - (x^2 + y^2)$, but the gradients point in opposite directions).
Another student volunteer will come up to draw a vector in the direction of ∇g. (I don't really care which way it points, as long as it is perpendicular to the level curve).

Step 8 shows examples of the gradient vectors they may have drawn. This step is included so that we can easily view a relationship between ∇f and ∇g. For this question, I want a student to chime in with the fact that the vectors are parallel. Then we recall that parallel vectors are just scalar multiples of each other. Aha! We have $\nabla f = \lambda \nabla g$ for some constant λ at each of the points.

Step 9: In this step, we visually verify that ∇f and ∇g are parallel at each of the 6 points of interest and nowhere else by moving a slider at the bottom of the screen to view the gradients at all the points along the circle. Thus, we have found a method for identifying these 6 high and low points! 1. Find the points where ∇f and ∇g are parallel and 2. stay on the constraint curve. This gives the 2 Lagrange equations: $\nabla f = \lambda \nabla g$ and $g(x, y) = c$.

Developed by Shelby Stanhope and Paul Seeburger for Taking CalcPlot3D to the Next Dimension, NSF – IUSE #2121152 and #2121133. These materials can be accessed at: https://sites.monroecc.edu/multivariablecalculus/3d-learning-activities/

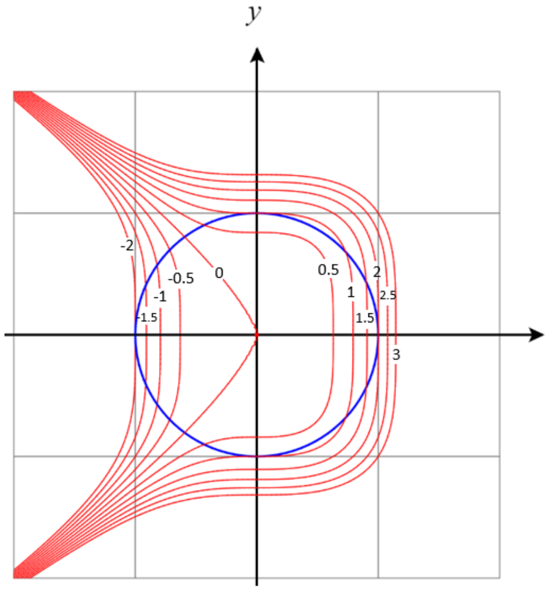
Page 2:



Developed by Shelby Stanhope and Paul Seeburger for Taking CalcPlot3D to the Next Dimension, NSF – IUSE #2121152 and #2121133. These materials can be accessed at: https://sites.monroecc.edu/multivariablecalculus/3d-learning-activities/

Practice Problem 1: Find the absolute maximum and minimum values of $f(x, y) = 2x^3 + y^4$ on the unit circle.

CalcPlot3D Link (move the b slider)
CalcPlot3D Link



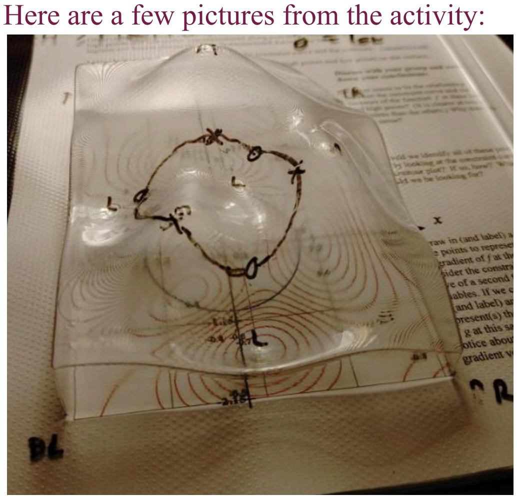
(I have students start this problem by marking the points where the constraint and the level curves are tangent/parallel on the contour plot provided. These are the points which will be output by solving the Lagrange equations. Then we set up the Lagrange equations and solve them. We evaluate the function at these points to identify the absolute max/min of the function. Then we view the CalcPlot3D links provided above. This helps to graphically solidify the concepts by showing the surface subject to the constraint with and without the level curves)

Summary: Method of Lagrange Multipliers

(Here we summarize the big picture of the method. 1. Set up the Lagrange equations, 2. Solve the set of Lagrange equations to obtain a list of points, 3. evaluate the function at all of the points and compare to find the absolute max/min of the function)

Practice Problem 2: A rectangular box without a lid is to be made from $12m^2$ of cardboard. Find the maximum volume of such a box.
(This provides an opportunity to solve a constrained optimization problem in the context of an application)

Here are a few pictures from the activity:



Developed by Shelby Stanhope and Paul Seeburger for Taking CalcPlot3D to the Next Dimension, NSF – IUSE #2121152 and #2121133. These materials can be accessed at: https://sites.monroecc.edu/multivariablecalculus/3d-learning-activities/