Contour Plot Activity Instructor Notes:

Contour plots are interesting because they are a 2D representation of a 3D surface (possibly given by a function of two variables, $z = f(x, y)$). Many concepts of multivariable calculus can be understood and assessed using contour plots, so it is important that students make a concrete connection between the spatial 3D surface and its 2D contour plot.

This activity starts by having students draw level curves onto a physical 3D model of a surface. Then, they draw these level curves onto a 2D graph on a whiteboard (or paper). Finally, they use CalcPlot3D to visualize both the 2D contour plot and the 3D graph, with the level curves on the surface.

Through the activity and guided worksheet, students will:

- Gain a concrete connection between the 3D surface and its 2D contour plot
- Connect the idea of contour plots to previous experience (e.g. topographic maps)
- Discover concepts relating contour spacing to surface steepness
- Foreshadow the gradient vector and its relationship to the level curves
- Interpret the meaning of a level curve in the context of an application problem
- Practice using CalcPlot3D to visualize contour plots

Materials:

- Scaffolded worksheet (Modify it however you wish for your class! We would love to hear about how you implement the worksheet, so please feel free to reach out to Paul and Shelby to share.)
- 3D models (3D-printed or thermoformed)
    - We suggest using the project mountainous surfaces (STL files can be accessed here) or you can choose to design your own surfaces.
    - We recommend one surface per 2-3 students.

- Whiteboard marker or wet erase marker to write on the model
    - 3D-printed model with no sealant: wet erase marker works fairly well, but have wet cloths on hand to erase if you teach back to back sections.
    - 3D-printed model sealed with XTC 3D Smoother: whiteboard marker
    - Thermoformed model: whiteboard marker
- Whiteboard or white sheet of paper to draw the 2D contour plot (could use back of worksheet)
- CalcPlot3D (via laptop, tablet, or phone). In general, the experience will be better on a laptop or tablet.
- Ruler (same number of rulers as models). This can be a real ruler or one printed on paper and cut out. The ruler is optional. Sometimes I forgo using the ruler and just tell students to estimate one cm. The learning opportunity provided by the ruler is that students must hold it vertically to measure the different z-values. Some students tend to hold the ruler diagonally against the mountain surface, which lends to a nice discussion opportunity about how we measure an elevation change from $z = 0$ to $z = 1$.

Activity Time: This activity has taken anywhere from 25-40 minutes depending on implementation style.

Developed by Shelby Stanhope and Paul Seeburger for Taking CalcPlot3D to the Next Dimension, NSF – IUSE #2121152 and #2121133. These materials can be accessed at: https://sites.monroecc.edu/multivariablecalculus/3d-learning-activities/

Activity Worksheet (With Instructor Notes)

Your group has a surface that is the graph of a function of two variables. The colored dot on the surface represents a point at which the elevation is zero. You also have a ruler and an erasable marker.

(Before class begins, I mark a point on the surface that will serve as the origin. I typically place this at a mid-level z elevation, since they will draw contours at both positive and negative z-values.

I typically tell students to get started by following instructions 1-4 below, without giving additional verbal instructions.)

1. Using the ruler and marker, mark the location of a mathematical bug at some point on the surface with $z = 1$ cm. Now, trace a trail for the bug to follow so that the bug remains at the same elevation. If you find that your bug has fallen off the edge of the surface or is starting to loop back around and retrace its steps, is there another place on the surface with elevation $z = 1$, where you might start in order to walk a new path? Attempt to walk all possible paths with elevation $z = 1$.

   (I often notice that students will draw only one level curve at $z = 1$, although there are other parts of the mountainous surface where the bug could walk at $z = 1$ (for example around a different hill). I ask these students leading questions to help them consider other parts of the contour..

   Some students mistakenly hold the ruler diagonally against the mountain surface, which lends to a nice discussion opportunity about how we measure an elevation change from $z = 0$ to $z = 1$.)

2. Do the same for a mathematical bug at elevations $z = 2$ and $z = 0$.

3. Now do the same for two other mathematical bugs at elevations of your choice. At least one of the elevations should be negative.

4. Next, go to the board with your surface and draw these various *level curves* on a 2D graph. Be sure to label the *level curves* on your graph with the appropriate *z*-values. Check the work of another group.

   (If you do not have whiteboard or chalkboard space, this can be done on a piece of paper, possibly the back of the worksheet..

   Students will often forget to label the curves with the elevations, which lends to a nice discussion about why these labels are needed. Given this 2D graph, how would I know which way is uphill/downhill?)

Question: Have you seen a graph of lines/curves like this before? If so, where?
(I hope for someone to mention hiking, elevation/ topographic maps here)

Concept Question: What do you think the term "level curve" means as it pertains to a function of two variables?
(Since the elevation is the same everywhere on this curve, we mathematically define that the function is equal to a constant, $f(x, y) = C$. I typically have this discussion as a class, so that we have mathematically defined a level curve.)

Instructor Demo: A 2D graph of the level curves of a surface is called a *contour plo*t. We can use CalcPlot3D to visualize contour plots.

(Here I give a brief demonstration of how to use CalcPlot3D to create a contour plot)

5. Now use CalcPlot3D to check the accuracy of your *contour plot* (Note that level curves are also called the *contours* of a surface). Depending on whether your surface is “white, green, or yellow”, the graph of your surface can be found in CalcPlot3D in the Menu →Examples → Function Surfaces→ Mountain White (Green, Yellow). Once you have pulled up the graph of your surface, generate a contour plot. Be sure to select thoughtful settings for your contours. This will depend on the actual *z*-values of the function, not the centimeter measurements we used above.

Question: Look at the steepest section on your surface. What corresponding feature(s) do you see on your contour plot? In general, how can we summarize how contour spacing relates to the slope of the surface?

> (We observe that the contours are close together when the surface is steep (you don’t have to move very far in the *xy*-plane to reach the next elevation), but the contours are further apart when the surface is less steep. This concept continues to be important throughout the course and comes up again many times. As an example, a problem on partial derivatives could include a contour plot and it could ask whether f_x is greater at two different marked points (that have different contour spacing in the x-direction).)

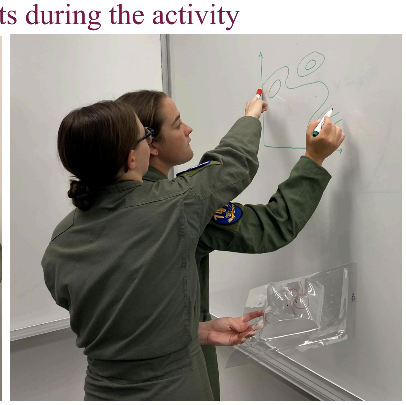
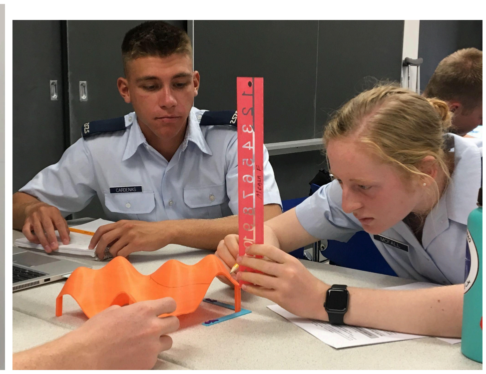
Question: Pick a point on your $z = 0$ level curve and assume the mathematical bug is at that point on the surface. Draw a vector at the point on the contour map on the board (not on your surface) that indicates the direction the bug should step in order to go uphill most rapidly at that point. Repeat this process for at least two additional points on this contour. What is the relationship between the vectors you drew and the level curve $z = 0$?

> (Here we are foreshadowing the gradient vector and its perpendicular relationship with the level curve. We will revisit this idea when the gradient vector is covered in a later section. I usually tell students this direction of greatest increase is going to be important to us soon!)

Question: Suppose a company determines its cost for producing a particular good based on the expenses of labor and materials. Let the function $z = C(x, y)$ represent the cost of producing the good, where x is the expense due to labor and y is the expense due to materials. Describe what the level curves of this function represent.

> (It is important to step back from the elevation interpretation of functions of two variables, and to discuss these concepts in a different setting. Here I hope that students can articulate that a level curve of this function represents all possible combinations of labor and expenses that result in the same production cost.)

Here are a few pictures of my students during the activity

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