# Dynamic Visualization Tools for Multivariable Calculus

## CalcPlot3D - A Guided Tour:

Contact info:     **Paul Seeburger**
**Monroe Community College**

pseeburger@monroecc.edu
http://web.monroecc.edu/calcNSF/

I.  First let's cover some of the basic functionality of the applet.

**Rotating the 3D graph:**
1.  Click on the 3D plot window and drag the mouse on the plot.  This should rotate the surface in the direction you move the mouse about the focus point (the origin, by default).  To reset the viewpoint to the original one, press the **Home** key (or choose **Reset 3D View** from the **View Settings** menu).

**Using 3D Glasses - Other 3D Viewing Modes**
2.  There are a variety of 3D viewing modes you can experiment with in CalcPlot3D.  These are found in the **Select 3D View** submenu of the **View Settings** menu in CalcPlot3D.  The standard view will always work, but sometimes it is fun to experience the 3D graphs with more realistic 3D depth.  If you have 3D glasses, there are a number of options (called Anaglyphs) you can try.  **Red**-**Cyan** or **Red**-**Blue** 3D glasses can be used for all Anaglyphs but the **Amber**-**Blue** Anaglyph option.  This one can only be viewed with **Amber**-**Blue** 3D glasses (distributed recently for the Super Bowl 2009 half-time special).  **Red**-**Green** 3D glasses can be used for the **Red**-**Green** Anaglyph option.  The other two 3D viewing options (**Stereo Pair** and **Cross-eyed**) do not require 3D glasses, but do take some practice.  The biggest benefit of these two modes is that you get a full color 3D experience without needing 3D glasses, but it is harder to share this view with a class full of students who may not have mastered one of these techniques yet.  For more information on these modes see the **3D View Help** on the **Select 3D View** submenu.

**Using the 2D Trace Plane:**

3. Click on the 2D trace plane on the left side of the applet and drag the cursor around this place. A trace point should appear on the 3D plot and move around the surface corresponding to the input point (x, y) that you specify in the trace plane.

4. Choose **Show the tangent plane at point** on the drop-down menu just above the trace plane. Then click and drag on the trace plane. You should see a tangent plane move around on the 3D surface.

5. Once the trace plot has the focus (by clicking on it), you can use the arrow keys to move the trace point left and right, or up and down. You can also choose to use scrollbars to move the point in the x- or y-direction only. This option can be found on the **Trace Plot** menu at the top of the applet. In most modes you can also right-click on the trace plot to see this option also.

6. Now hide the tangent plane (and the trace point) by selecting **Show graphs only** on the drop-down menu above the trace plane. As you can see, there are more options on this menu to experiment with later. You can also enter a trace point's coordinates (or a direction vector for the directional derivative option) using the **Trace Menu**. Click on the **Graph** button to reset the 3D view.

**Contour Plots:**

7. Select the **Draw Contour Plot** menu option from the **Contour Plot** menu at the top of the applet. This will open up a new dialog box that looks like this:

Note that you can select which function to create the contour plot for. You can also choose the first level value, the step size, and the number of contours to create. This will create the contour plot using equally spaced contours. If you prefer, you can also create your own list of level values using the Enter List button. The list of level values is shown along the bottom of the dialog box.
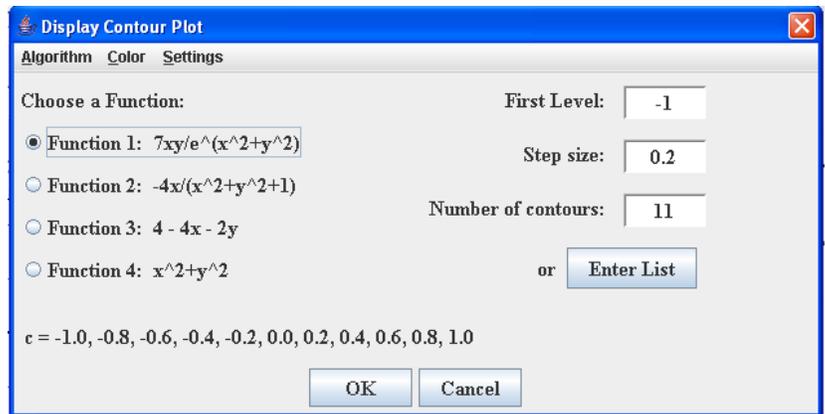


Let's create a contour plot for function 1. Change the **First Level** to -1.2 and the **Number of contours** to 13. Then click on the **OK** button. Immediately the contour plot should begin to be drawn.

8. Once the contour plot is drawn, follow the instructions given above the 3D plot and click anywhere on the contour plot. This will cause the 3D surface to be redrawn with the contour plot shown with it in 3D. (Note that this assumes you had the surface corresponding to Function 1 graphed before creating the contour plot. If not, you will get the 3D version of the contour plot alone or with some other surface which was plotted before the contour plot was created).

To make this easier to see, note that the surface is automatically drawn without the wireframe edges and using a semitransparent mode, letting us see through the surface to see the contours more clearly. To show the edges again, you can select **Show Edges on Surface** from the **View Settings** menu at the top of the applet (or you can simply hit the E key to turn edges back on or off again).

9. Use the **Reset Applet** menu option on the Graph menu at the top of the applet. This should reset the applet to its original configuration.

**Graphing Functions of Two Variables:**
10. Now let's try some other functions of two variables.  To graph other functions, you can either select a function from the drop-down list or you can enter your own function in the function textbox.  Examples of most of the supported functions are in the list, but some additional functions are supported, including the hyperbolic trig functions.

11. Try graphing the function, $z = \arcsin(x + y)$.  Notice that it appears to be a narrow ribbon.  Press Control-Home.  This changes the viewpoint so that you are seeing the surface from above the $xy$-plane (from the positive $z$-axis).  Now press the **P** key (or choose **Change to Parallel Projection** from the **View Settings** menu).  This takes away the 3D perspective and allows you to look at the approximate natural domain of this function of two variables.  As you can see, this domain is clearly the region between the lines $y = -x - 1$  and  $y = -x + 1$.

12.  Now turn Perspective Projection back on (either press the **P** key again or use the menu option). Then try the function,  $z = 1/(y - x^2)$.  As you can see from this function, CalcPlot3D does a decent job at correctly graphing discontinuous surfaces.  There are still some exceptions, but it handles most common types of discontinuity.  Try some other discontinuous functions to see what its limits are.  Another nice example of a discontinuous surface in the function list is:  $z = (2x - y^2)/(2x^2 + y)$.  For either one of these, try viewing the natural domain as you did in step 11.  Note that it appears that more is missing than you might expect.  This is because of the min and max $z$ cutoff values.  Open the Format Axes dialog by either typing the **A** key or choosing the **Format Axes** option on the **View Settings** menu.  Now change the **zClip-bot** value to -40 and the **zClip-top** value to 40 (-20 and 20 work almost as well).  These two values determine the bottom and top z-levels at which to "clip" the surface for clearer viewing.  Changing these values to -40 and 40 will fill in more defined points and show the missing parabola from each function's domain more clearly.  When you are ready to continue, change the two zClip values back to what they were originally or simply re-enter one of the other window values and they will be automatically reset to the appropriate default values.

For each of the four functions of two variables, there are two ways to change the type of function to graph.  You can either right-click on the "**z =**" in front of the function or you can select **Set Function Type** with the function number from the **View Settings** menu.  Options include:  $z = f(x, y)$,  $y = f(x, z)$, or  $x = f(y, z)$ for functions in Rectangular Coordinates,  $r = f(\theta, z)$  for functions in Cylindrical Coordinates, and  $\rho = f(\theta, \varphi)$  for functions in Spherical Coordinates.

**Graphing Functions in Cylindrical Coordinates:**
13.  Select the Cylindrical Coordinate type,  $r = f(\theta, z)$ for Function 1.  Initially the default function is  r = 2, which graphs a cylinder of radius 2.  Now select a couple of different options from the drop-down menu.  These are updated to reflect the function type selected.   For example, select the function, $r =$ sqrt($z$).  This creates a particularly nice looking graph of the paraboloid, $z = x^2 + y^2$.

When one of the four main functions is cylindrical, the **Cylindrical Function Parameter dialog** is displayed.  You


Cylindrical Function Parameters (Click on θ below to paste it into the function window.)
Function 1 ▼   0   ≤ **θ** ≤   2pi   #Steps:  20      0   ≤ z ≤   pi   #Steps:  20

can set different ranges of $\theta$ and $z$ and number of steps to use for each cylindrical function.   You can also use the $\theta$ button to insert $\theta$ in your functions.  Typing **theta** in the function textbox will also work.

Try increasing the number of steps for $\theta$ to 30 steps.  Press enter or click on the **Graph** button again.  Next increase the  $z$  range to go from 0 to 5.  Again, to see its effect, press enter or click on the **Graph** button.

14. On the Trace Plane select the first mode: Show a trace point on the surface. Note that this now allows you to vary $\theta$ and $z$ and see the location of the point on the cylindrical surface!

**Graphing Functions in Spherical Coordinates:**
15. Now select the Spherical Coordinate function type, $\rho = f(\theta, \varphi)$, for function 1. Similar to the cylindrical option above, the default function is $\rho = 2$. This time we get a sphere of radius 2. Try selecting a few of the functions from the Function 1 drop-down menu. For example, $\rho = \theta * \cos(\varphi)/2$ produces an interesting shell-like surface.

As with the cylindrical option above, there is a **Spherical Function Parameter dialog** that displays when at least one of the four functions is spherical. As before, you can vary the range and number of steps for the two variables, $\theta$ and $\varphi$. Also as before, you can use the $\theta$ button or the $\varphi$ button to insert $\theta$ or $\varphi$ into your functions. Typing **theta** or **phi** in the function textbox also works.

**Using Parameters:**
16. a. Type the following functions into the Function 1 and Function 2 textboxes without pressing enter or updating the graph in between: $z = \text{sqrt}(1 - x^{\wedge}2 + y^{\wedge}2)$ and $z = -\text{sqrt}(1 - x^{\wedge}2 + y^{\wedge}2)$. Be sure both functions are selected using the checkboxes next to them. Then click on the **Graph** button or press enter from one of the textboxes. You should see a hyperboloid of one sheet (assuming you have the default zoom of -2 to 2 on the x- and y-axes). This is a level surface of the function, $f(x, y, z) = x^2 - y^2 + z^2$, with $x^2 - y^2 + z^2 = C$ and $C = 1$.

b. Next edit the two functions (again without pressing enter or updating the graph in between), replacing the 1 with the letter **C**. Once both are in place, press enter or click on the **Graph** button.

You will see a new dialog window appear titled **Adjust Parameters**. Use the scrollbar to the left of **c = 1** to vary the value of this parameter and watch the hyperboloid morph through various level surfaces.

c. Deselect the checkbox next to a = 1 and select the checkbox next to the c = 1. Now click on the Animate Parameters button. You will see the value of the parameter C is automatically varied through its range of values forward and then backward until you click on the same button again (now labeled, Stop Animation). You can vary the speed by either entering a number larger than 1 or by using the up/down arrow keys or the plus/minus keys while the parameter is being animated.

You can use up to four constant parameters (a, b, c, and d) in functions of two variables, implicit surfaces, parametric surfaces, space curves, or vector fields. Experiment with them and let me know what uses you find for this feature. It allows us to ask a lot more "what-if" questions!

**Graphing Space Curves:**
17. Before we graph a space curve, let's use the **Reset Applet** menu option on the **Graph** menu at the top of the applet to reset the applet to its original configuration. Then use the **Delete** key or choose **Clear All Graphs** from the **Graph** menu to clear the graphs.

18. Now choose **Add a Space Curve** from the **Graph** menu. Click on the **Graph** button at the bottom of the Space Curve dialog (shown on the right). A multicolored helix should appear on the 3D plot. The color variation along the curve is meant to help viewers see the curve's 3D features, but space curves also really benefit from the other 3D modes. Try using whichever of the 3D modes
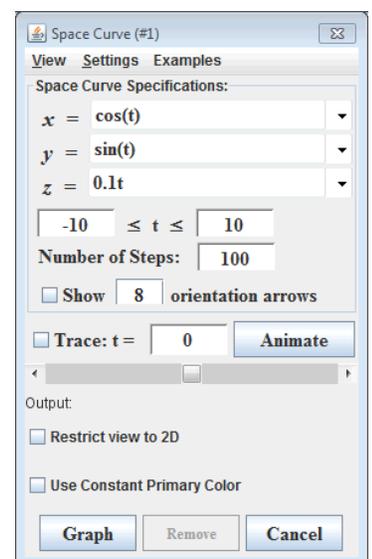
was easiest to view when we experimented with them earlier. Note that you can choose to select **Use Constant Primary Color** just above the **Graph** button on the Space Curve dialog, if you would like to see the curve in a constant color like red. Constant color is still the default for 2D curves, although you can select other color modes for these curves as well.

19. Click on the **Animate** button. This should cause a blue vector to appear which traces out the space curve. This is the vector given by the vector-valued function, $\bar{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$.

20. Next select the **View** menu at the top of the Space Curve dialog. Note that you can also select to show the velocity and/or acceleration vectors at the point in space, assuming the original vector-valued function stated above is a position function. Select both of these so that all three vectors will be shown and then click on the **Animate** button again. As the curve is traced out, notice the velocity vector (tangent to the curve in the direction of motion) and the acceleration vector (in this case always pointing toward the $z$-axis in the center of the helix.

21. Now select the first choice on the **Examples** menu at the top of the dialog. It is called **Coil Ring**. Then click on the **Graph** button. You should see a circular coil. Use the **View** menu to deselect the velocity and acceleration vectors and to select the TNB-frame with all features (the TNB labels, and the three associated planes). Click on the Animate button again (or use the scrollbar just below it) and watch how the TNB-frame varies as you move along this curve.

22. Select the Spirograph option on the **Examples** menu. Note that $z = 0$. Note also that the checkbox is selected next to **Restrict view to 2D** just above the **Graph** button on the Space Curve dialog. This option allows you to explore plane curves in a more focused, easier to see way. As its name implies, this curve reminds me of something created with Spirograph (a pattern drawing "game" from when I was a child). Deselect the options we selected above and select the **Osculating Circle** and the **Curvature value**. Use the scrollbar to vary the position of the point along the curve and observe how the osculating circle and the curvature value vary. Note what happens at the sharp turns!

When you are finished with the 2D plane curve above, be sure to deselect **Restrict view to 2D**. Also clear the graph by either choosing **Clear All Graphs** from the **Graph** menu or pressing the **Delete** key.

23. Here is a hands-on visual homework check I have my students do as part of one of their graded assignments. Try it out!

    a. Determine the line of intersection of these two planes. Write the parametric equations for this line, showing all work. $2x + y - 2z = 5$   &amp;   $3x - 6y - 2z = 15$

    b. Use the CalcPlot3D applet to display these two planes. You will have to solve each planar equation for $z$. Zoom out a couple times until you can see both planes along with their intersection. Now also show the line of intersection. (To do this, choose **Add a Space Curve** from the **Graph** menu, and enter the parametric equations for the line.) Verify that your line is at the intersection of the two planes. Rotate until you have a good view of the two planes and their line of intersection. Print out your resulting figure and hand it in with this assignment.



| Answer Line: | $x = 3 + 14t$ | Answer line graphed with these planes: |
|---|---|---|
| | $y = -1 + 2t$ | $z = (2x + y - 5)/2$ |
| | $z = 15t$ | $z = (3x - 6y - 15)/2$ |
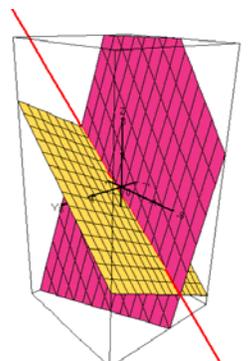
**Graphing Points & Vectors:**

24. (Graphing the plane through three non-collinear points.) First, clear the 3D plot by either choosing **Clear All Graphs** from the **Graph** menu or pressing the **Delete** key.

    a.  Find the equation of the plane containing the points (2, 0, 1), (-1, 2, 3), and (0, 2, -2).

    b.  Graph this plane along with the three points to verify that all three points lie on the plane. To do this, first solve the plane equation for $z$ and graph the plane. Then select **Add a Point** from the **Graph** menu, and enter the coordinates of one of the points. Then select the size of the point to be 4. Repeat these steps to add the other two points. Rotate the plot to verify that the points lie on the plane.

    Answer:  Plane equation is:  $z = (22 - 10x - 13y)/2$

25. (Graphing a 2D gradient vector of a function of two variables.) Clear the 3D plot by either choosing **Clear All Graphs** from the **Graph** menu or pressing the **Delete** key.

    Find the gradient vector of the function $z = x^2 - y^2$ at the point (2, -1) and graph it along with the surface and the point on the surface above (2, -1). Graph this gradient vector twice. Once with its initial point at the origin and again with its initial point at (2, -1, 0). Make the first one blue and the second one red.

    To do this, first graph the surface. Then graph the point on the surface above (2, -1). Here this is the point (2, -1, 3). Then to add the vectors, use the **Add a Vector** option on the **Graph** menu. Enter the vector <4, 2, 0> in the top textbox. Next to initial point, first enter the origin (if it is not already there). Then choose the color from the palette. Repeat this to graph the vector again, this time with its initial point at (2, -1, 0). Note that this vector gives the compass direction you would need to move toward in order to ascend most steeply from the given point on the surface.

26. (Graphing a 3D gradient vector of a function of three variables.) Clear the 3D plot by either choosing **Clear All Graphs** from the **Graph** menu or pressing the **Delete** key.

    Find the gradient vector of the function, $f(x, y, z) = x^2 + y^2 - z$, at the point (1, -1, 2) and graph it along with the level surface, $f(x, y, z) = 0$ and the point, (1, -1, 2).

    To do this, first graph the level surface, $z = x^2 + y^2$. Next graph the point (1, -1, 2) which should be on the surface. Finally, graph the vector, making its initial point (1, -1, 2). Note that the gradient vector is normal to the level surface at this point!

**Graphing Vector Fields:**

27. Clear the 3D plot by either choosing **Clear All Graphs** from the **Graph** menu or pressing the **Delete** key. Select **Add a Vector Field** from the **Graph** menu. Click on the **Graph** button at the bottom of the Vector Field dialog. This will create a 2D vector field (on the 3D plot) in the $xy$-plane. As you can see in the figure below, leaving P blank allows you to graph 2D vector fields. If we enter an expression for P, then the number of levels given in the $z$-direction is specified by the bottom textbox. The default is set to 4. The default scaling option, divides all vector magnitudes by 4 to make them easier to distinguish from each other in the vector field plot.

28. Press **Control-Home** to view the vector field from above the xy-plane. (You may need to click on the 3D plot first.) Then press the **P** key to switch to Parallel view. This will give a true 2D vector field plot.

29. Experiment with the Mode options. Rectangular Array is the default configuration, but Cylindrical and Spherical Arrays are also available. Both of these last two modes give identical results in a 2D vector field. They give a circular array of vectors rather than a rectangular array.

30. Now press **P** again to return to the 3D perspective mode and press **Home** to view the plot from a 3D viewpoint. Let's experiment with some 3D options.

31. First enter P = z. Then click on the **Graph** button in the Vector Field dialog. Try all three modes. My favorite 3D vector field mode is Spherical Array, although I generally prefer the Rectangular Array mode for 2D vector field plots. Try using one of the 3D Anaglyph modes with 3D glasses or the Stereo Pair mode. In the same way that these 3D options enhanced the viewing of the space curves we discussed earlier, I find the 3D vector fields are much more impressive this way. You may even find it helpful to set the color of the whole vector field to blue or some other color, even in the standard 3D mode. By default vectors are colored based on their direction in three space.
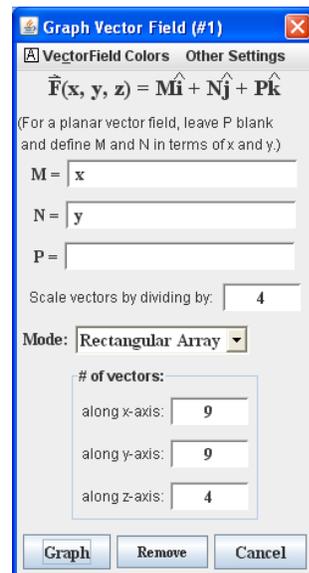
**Viewing the gradient vector of a function of two variables with the corresponding surface and gradient vector field:**

32. Clear the 3D plot by either choosing **Clear All Graphs** from the **Graph** menu or pressing the **Delete** key. Then graph the function, $f(x, y) = x^2 + y^2$.

33. Now add this function's gradient vector field to the plot.
   Hint: It is $\bar{\nabla}f(x, y) = 2x\,\hat{\mathbf{i}} + 2y\,\hat{\mathbf{j}}$. Note that you need to make P blank or make just 1 layer of vectors along the z-axis in order to create a 2D vector field.
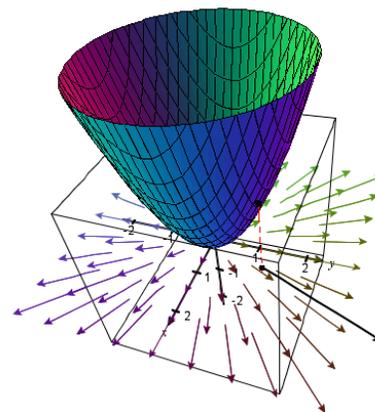
34. Now select the **Show the gradient vector at point** option on the drop-down menu below the trace plane on the left side of the applet. Click and drag on the 2D trace plane and you will see the gradient vector moving around inside the gradient vector field, and the corresponding point on the surface. Note that the gradient vector from the trace plane is full size, while the gradient vector field vectors are likely scaled (unless you have set this scale value to 1).

Note that you can also press **Control-Q** to switch between showing the gradient trace vector at the point in the xy-plane and at the actual point on the surface.

**Showing gradient vectors on contour plots:**
35.  First create a contour plot for a function of two variables like the one we just graphed above.

36.  Then before you click on the contour plot, select the **Show the gradient vector at point** option on the drop-down menu below the trace plane (if it is not already selected).  Click and drag on the 2D trace plane and you will see the gradient vector moving around on the contour plot.  This makes it possible to show visually that the gradient vectors are always normal to the contours of a function of two variables.

**Graphing Regions (for visualizing double and triple integrals):**
37.  Clear the 3D plot by either choosing **Clear All Graphs** from the **Graph** menu or pressing the **Delete** key.  Also select **Show graphs only** from the trace plot drop-down menu.
Then graph the function, $f(x, y) = 2 - x^2 - y^2$.

38.  We will create the region whose volume is equal to the double

integral: $\int_{-1}^{1} \int_{x^2}^{1} (2 - x^2 - y^2)\, dy\, dx$.  See the figure to the right.
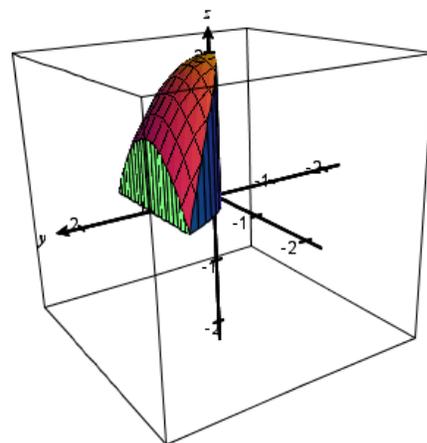


Select **Add a Region** from the **Graph** menu.
Now select Function 1 to the upper right of the **Create Bounded Region** dialog and select **Show Region only (Clear Plot 1st)**.  As you can verify, the default region in the *xy*-plane is the one we desire for this example, so we will leave it alone for now.  Click on the **Graph Region** button at the bottom of the **Create Bounded Region** dialog.

Note:  If you want to modify the region again (once the view is cleared), it is more efficient to deselect the checkbox labeled, **Show Region only (Clear Plot 1st)**.

39.  Let's do this again to obtain the region whose volume equals the double integral:
$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (1 + \sin(x^2 + y^2))\, dy\, dx$.

First graph the function, $f(x, y) = 1 + \sin(x^2 + y^2)$.

Then select **Add a Region** from the **Graph** menu, or simply edit the dialog you opened earlier.
Modify the top function to be y = sqrt(4 - x^2) and the bottom function to be y = -sqrt(4 - x^2).  You can either enter these by hand or select them from the respective drop-down lists.  Next change the xmin and xmax to -2 and 2, respectively.  Then click the **Graph Region** button.  This double integral can be integrated using polar form and the region looks like a jello from a mold.

Note that you can enter the region in the *xy*-plane using functions of *y* as well.  It is also possible to add rectangular prisms to form an inner or outer partition of the given region in the *xy*-plane.  You can even select to use scrollbars to vary the number of subdivisions to use when creating the partition.  (It is also possible to use the whole *xy*-plane viewing region to create a Riemann partition of any rectangular function using the **Show Riemann Prisms** option on the **Tools** menu.)

You can now select the **Polar** tab in the lower left section of the **Create a Region** dialog.  You may then specify any polar region in terms of r and theta.  Polar prisms can then also be displayed and varied.

40.  For visualizing triple integrals, we can use this Region feature to create the 3D region of integration for a number of triple integrals.  Eventually it will be possible to specify a lower function as well as an upper function in the *z*-direction which will give much more flexibility to this option, but for now it is possible to give some simple examples of this nature.


**Graphing Parametric Surfaces:**

41.  Clear the 3D plot by either choosing **Clear All Graphs** from the **Graph** menu or pressing the **Delete** key.

42.  Now select **Add a Parametric Surface** from the **Graph** menu.  A dialog box will be displayed on the right side of the screen.  You can enter any parametric surface using this dialog and using parameters u and v.  Try clicking on the **Graph** button to see the default parametric surface.  This one looks a little like a fat pillow.

43.  Now choose the **Examples** menu at the top of the dialog and explore the example surfaces shown there.  In particular be sure to see the Moebius Strip, the Trefoil Knot, the Big Knot, the Klein Bottle, and the Klein Figure 8 Surface.  The knots are particularly impressive when viewed using one of the 3D glasses options.  Try it out!
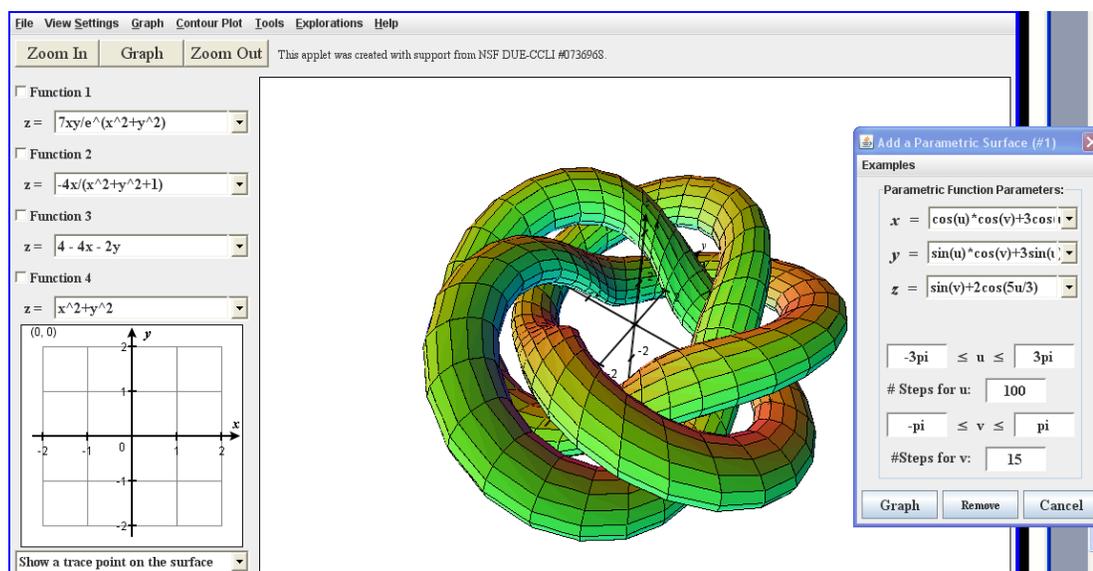


## II.  Exploring Other Features and Concepts

**Modifying color scheme of surfaces for increasing contrast between multiple surfaces graphed at the same time:**

When you graph two or more related surfaces at the same time, it is sometimes useful to change the color scheme of one or more of the surfaces to increase the contrast between them.  This can be done using the F1, F2, F3, and F4 keys.  These function keys change the color scheme of Function 1, Function 2, etc., respectively.   You can also format each surface using the **Format Surfaces** menu option on the **View Settings** menu.

The cycle of color schemes is:  standard, wireframe, reversed color scheme, transparent, transparent with a colored framework

**Taylor Polynomials of a function of two variables:**

This topic is something that we cover at my college as part of our objectives up to the second degree (quadratic) Taylor approximation of a function of two variables. I had always wanted to generalize this process and see what the results would look like, and eventually I was able to implement this feature. The applet will allow you to use a scrollbar to scroll through the Taylor polynomials of a function of two variables for any rectangular function of two variables it is given. The applet attempts to compute up to the 15$^{th}$ degree Taylor polynomial, but may reach only a much lower degree if the function's partial derivatives are fairly complex. Let's try it out!

1. First clear the 3D plot by either choosing **Clear All Graphs** from the **Graph** menu or pressing the **Delete** key.

2. Graph the function, $f(x, y) = \text{Cos}(x)*\text{Sin}(y)$. Then zoom out to -4 to 4 in the $x$ and $y$-directions.

3. Now select the **View Taylor Polynomials** option from the **Tools** menu at the top of the applet. It will take a few seconds as the computer calculates the partial derivatives and creates the Taylor polynomials. This example is successfully calculated all the way up to the 15$^{th}$ degree polynomial. Once it is ready, the original function is graphed as a wireframe and the 1$^{st}$ degree Taylor polynomial (the tangent plane) is shown. A scrollbar appears along the bottom edge of the 3D plot. Use this scrollbar to scroll through the various Taylor polynomials of this function. Note that only odd degrees add new terms for this particular function. As you increase the degree of the Taylor polynomial notice how the polynomial of two variables fits the original surface better and better around the origin until it is a fairly good approximation of the whole visible surface at the 15$^{th}$ degree.

4. To better view the Taylor polynomial itself (shown in the text window just above the 3D plot), you can click and drag on the equation and view all terms, dragging the equation left and right. You can also use the **Tools** menu option **Use Factorials in Taylor Polynomials** to switch this property on or off. Using factorials makes the form of the terms of the higher order Taylor polynomials easier to see, and the terms also generally take up less horizontal space each.

$$P_4(x, y) = y - \frac{x^2 y}{2} - \frac{y^3}{6}$$

Taylor polys centered about pt: (0, 0)

5. You can also vary the center point for the Taylor expansion using the Tools menu option just below View Taylor Polynomials. The default center point is the origin.

6. Other nice functions to try centered about the origin include:

   - $f(x, y) = \text{Cos}(x) - \text{Sin}(y)$
   - $f(x, y) = \text{Sin}(2x) + \text{Cos}(y)$
   - $f(x, y) = \text{Sin}(x^2 + y^2)$
   - $f(x, y) = x*e^\wedge y + 1$
   - $f(x, y) = e^\wedge(x^2 + 2x - y)$
   - $f(x, y) = \arctan(x*y)$
   - $f(x, y) = \arctan(x+y)$
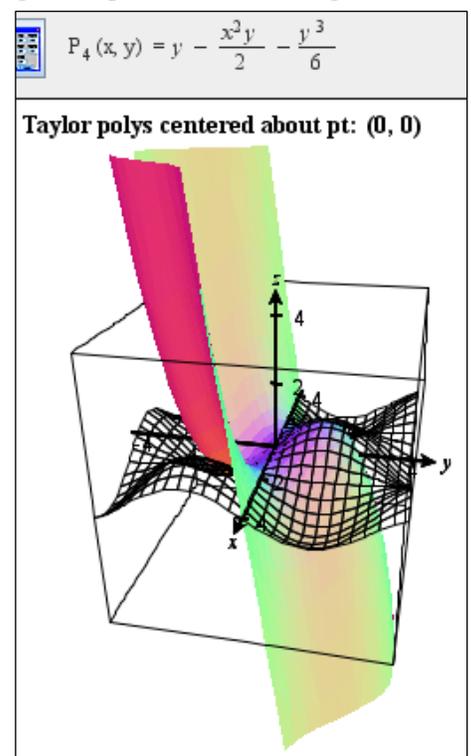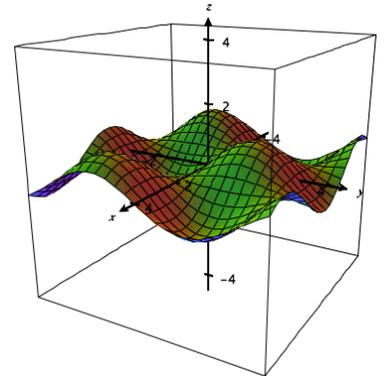
**Visualizing Lagrange Multiplier Optimization:**

When trying to visualize Lagrange multiplier optimization, it is helpful to be able to see the constraint on the contour plot. This is easy to do using the **Add Constraint Curve** submenu on the **Contour Plot** dialog. Let's explore this feature by illustrating the following problem:

Find the relative extrema of the function, $f(x, y) = x^2 + 2y^2$, subject to the constraint, $x^2 + y^2 = 1$.

1. Reset the applet by either restarting the applet or using the **Reset Applet** menu choice on the **Graph** menu.
2. Enter the function z = x^2 + 2y^2 as Function 1. Graph it to see this elliptic paraboloid.
3. Select **Draw Contour Plot** from the **Contour Plot** menu.
4. Modify the parameters to start at level c = 0, use 0.25 as the step size, and use 17 steps.
5. Now use the **Add Constraint Curve** menu to choose **Circle centered at origin**. Click OK to accept the value of 1 for r². Note the other options available for creating constraint curves.
6. Click OK again on the Contour Plot dialog to create the contour plot and the constraint curve.
7. Now notice that a scrollbar has appeared at the bottom of the 3D plot window. This scrollbar can be used to move a point along the constraint curve through the contour plot. The coordinates of the point on the trace of the function *f* generated by this constraint are shown in the information panel above the 3D plot. Note there are also some instructions here.
8. By default, the gradient vectors of *f* and of the constraint (considered as a level curve of a function *g*(*x*, *y*)) are shown at this point as well.
9. As you move this point along the constraint, watch to see how the gradient vectors become parallel whenever the value of *f* is at a relative extremum subject to the constraint.
10. Now click on the contour plot itself. This will render the contour plot and the trace of the constraint on the 3D surface plot. Use the scollbar to move a point along this trace curve and watch to see where you obtain relative extrema. You can still see the gradient vectors on the small contour plot to the left.
11. If you want the actual constraint curve to remain visible in the xy-plane when you go to the 3D view, you can select this option on the **Add Constraint Curve** menu. Try it and then use the scrollbar to move the point in the 3D plot again. You should see the point on the constraint curve in the xy-plane as well as the corresponding point on the surface trace above(or below) this point in the domain.
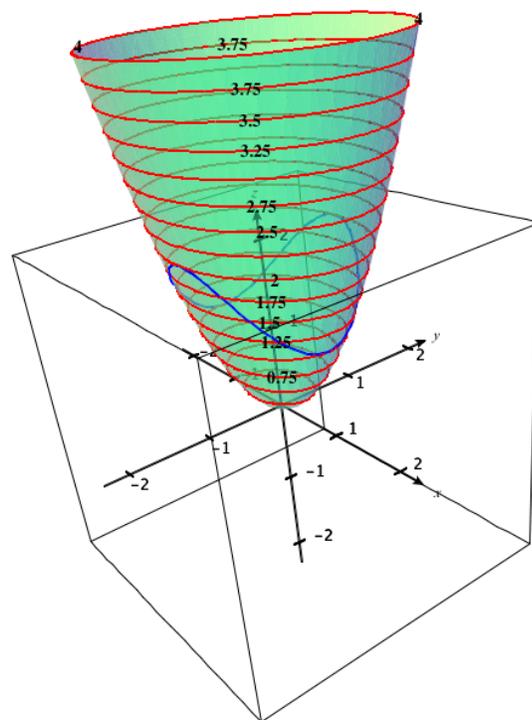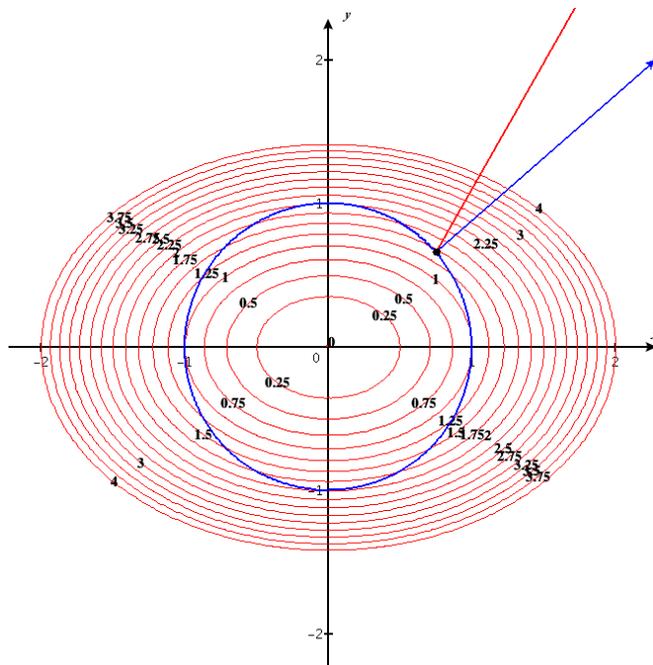
**Coloring a Surface using a function of three variables, $f(x, y, z)$:**

It is possible to define the colors of the surface using a function of three variables, ,$f(x, y, z)$. The colors represent values of the function at points on the surface. These values may represent temperatures, densities, intersections with particular level surfaces or perhaps z-heights.

1. Reset the applet by either restarting the applet or using the **Reset Applet** menu choice on the **Graph** menu.
2. Next select Use a function f(x, y, z) to set all surface colors from the View Settings menu (or use Ctrl-w). When asked for a function, use the default: $f(x, y, z) = z$. This colors the surface by z-level. Use the default blue to red coloring scheme. When asked for a lower function value to associate with blue, enter -1. Enter 1 for the upper function value to associate with red. These values will depend on the visible range of the function you are viewing. Values that are larger than the upper value will fade to white, while values lower than the lower value will fade to black.
3. Now try $f(x, y, z) = x^2 + y^2 + z^2$. This looks particularly nice with a series of concentric spheres defined using spherical coordinates. This coloring function makes the spheres vary in color by level. For the spheres, you will want to set the lower function value (blue) to 0 and the upper function value (red) to your highest radius.
4. If the surface is defined using the real part of a complex function, $z = f(w)$, then you may wish to define the color function in terms of $w$. The default coloring function for a complex function is Arg($f(w)$), although you can change this to any function of x, y, z or any function of w. The default color scheme for this mode is to use the full spectrum color wheel and use values from $-\pi$ to $\pi$.


**Graphing Complex Functions over the Complex Plane:**

You can graph complex functions in the form, $z = f(w)$, over the complex plane. By default the real part is assigned to the z coordinate, but you can choose to graph the imaginary part by using the Im function, e.g., $z = \text{Im}(w^3)$. You can also use the Arg or the Abs function to assign a real value to the z coordinate.

1. To be able to enter a complex function, you can either right-click on the "z=" to the left of Function 1 and select this type of function from the list of options, or you can use the Function Type submenu on the **View Settings** menu to do this.
2. The default function is $z = |w − d|$, where w is the imaginary variable varying across the complex plane and d is a complex parameter, which we can vary. There are currently four available complex parameters for use in these functions: a, b, c, and d. They appear on the complex Trace Plot in the lower left corner of the applet, and they can be moved around the complex plane to show how this affects the corresponding surface(s). You can also set them to particular complex numbers by either right-clicking on one and entering a value or by selecting this option on the **Trace Plot** menu at the top of the applet.
3. Try dragging the complex parameter point d around on the complex trace plane. You should see the cone's vertex move around to follow this point.
4. Now select a different complex function from the list or enter one of your own. Experiment a little.
5. Next select the function, $z = |(w^2 + 1)/(w^3 − 2)|$. This gives an interesting function that can be thought of in terms of sinks and poles.
6. Let's color this surface using the Arg$((w^2 + 1)/(w^3 − 2))$. This is the default coloring function for this complex function. Let's turn this coloring feature on by choosing this option on the View Settings menu. Note you will be asked to verify that you want this coloring function, and asked a

few more questions. Use the defaults this time. They are different defaults for a complex function than we saw above for a regular function of two variables.

7. Now let's view this surface from above. Hit Ctrl-Home, and then hit the P key to make the view parallel instead of perspective. This will give us an interesting and colorful view of the behavior of the function around the poles and sinks. This view can be improved by increasing the grid size and/or by using the gradient color mode (Ctrl-g turns this mode on/off).

**Graphing Implicit Surfaces (Implicit Equations in 3 variables):**

You can graph any implicit equation of the form, $f(x, y, z) = c$ or $f(x, y, z) = g(x, y, z)$. For example, you can graph spheres or quadric surfaces by entering their equations directly.

1. Reset the applet by either restarting the applet or using the **Reset Applet** menu choice on the **Graph** menu.
2. Select **Add an Implicit Surface** from the **Graph** menu.
3. The equation of a hyperboloid of one sheet is displayed as the default implicit surface. Hit the Graph button. You should see the hyperboloid appear in the 3D plot.
4. You can enter any equation involving $x$, $y$, and $z$ (Cartesian coordinates), any equation involving $r$, $\theta$ and $z$ (Cylindrical coordinates), or any equation involving $\rho$, $\theta$, and $\varphi$ (Spherical coordinates).
5. Try entering a few equations in each coordinate system. Note that you can either type the words rho, phi, and theta, or you can use the template buttons for these variables shown on the dialog.
6. Now you can vary the accuracy of the plotting algorithm by moving the scrollbar. This changes the number of cubes in each direction. The higher this number, the more accurate the surface, but also the longer it takes to display (and the more memory it uses). Note that you do need to keep this value within reasonable limits (less than 100 or so) or the applet will run out of memory.
7. Now let's play with the **Examples** menu. There is a large collection of examples to explore. Select the surface labeled, **Pipe Connector (Schwartz Surface).** The first part is my nickname for this surface. This is a nice example of an implicit surface that goes beyond quadric surfaces.
8. Now select the **Blended 3 Cylinders**. Note that it is formed by multiplying the expressions for each cylinders together to a product of 1.
9. Now select the **Linked Tori**. This example shows how we can get the union of multiple surfaces. Just solve each equation for 0 and multiply these equations together to form a product of 0. This effectively merges the various surfaces together. Other examples of this can be found on this list, including **Six Rings** and **Double Sphere** (on the **Test Cases** menu).
10. Note there are also examples of implicit equations using cylindrical and spherical coordinates.
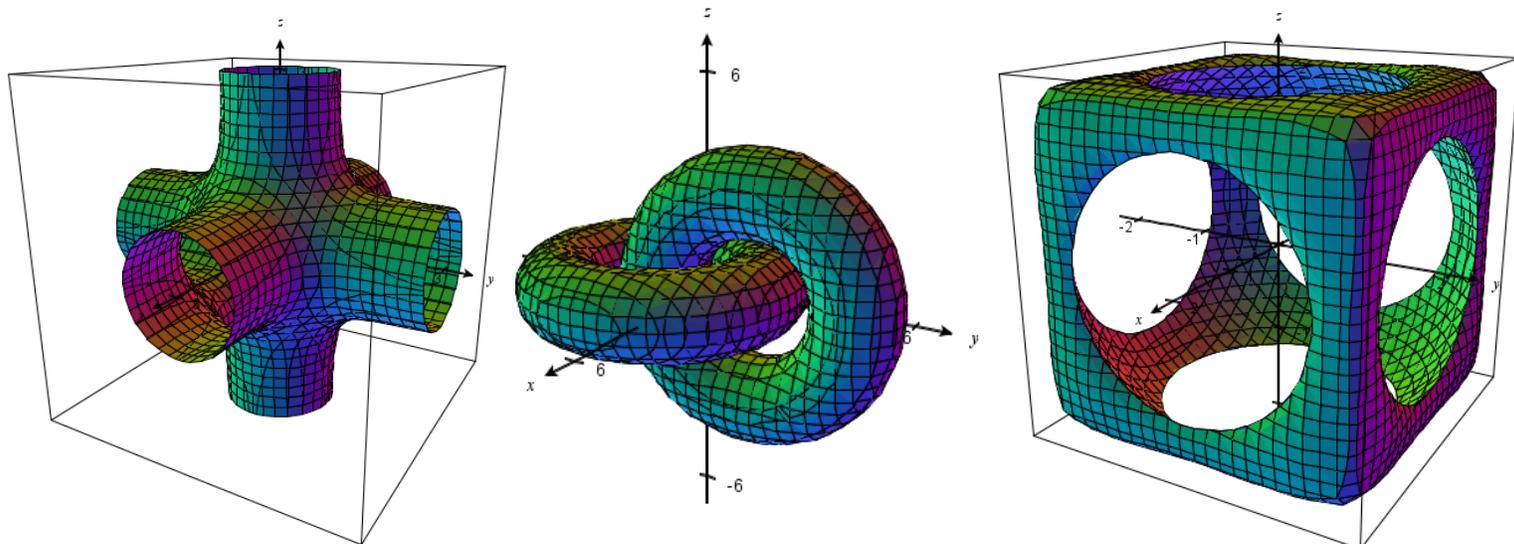
**More Parameter Examples:**

As mentioned above (on page 3), you can now use value parameters a, b, c, and d in functions of two variables, space curves, implicit surfaces/equations, parametric surfaces, and vector fields. Once you graph any of these objects using one or more of these parameters, you will see the Adjust Parameter dialog window appear. Using the corresponding scrollbars or textboxes, you can vary the values of each parameter and watch the effect this has on the surface, curve, or vector field. You can also use the **Animate Parameter** button to animate this morphing process like a little movie. On page 3 above, we described how to use this feature to illustrate level surfaces using the function textboxes. Here we will explore two more examples of using parameters.

**Example 1, Level Surfaces:**
1. Clear the 3D Plot using the **Clear All Graphs** button. Then select **Add an Implicit Surface** from the Graph menu.
2. Graph the given default implicit equation, x^2 - y^2 + z^2 = 2. This is a hyperboloid of one sheet. Now change the 2 to an **a** (or another of the four value parameters). Press enter or click on the Graph button. The Adjust Parameters dialog should be displayed at the upper left corner of the screen.
3. Using the scrollbar corresponding to the parameter you chose to use in the implicit equation, vary the parameter's value and watch how the level surfaces change in real-time.
4. Now click on the Animate Parameter button. The level surfaces of this function of three variables will now be shown as a little "movie", progressing through the specified range of values for the parameter first from smallest to largest and then from largest to smallest value in this range.
5. On the Adjust Parameter dialog, click on the tab labeled Range. You can now adjust the values of the range for each parameter, and specify the number of steps to use. This will be especially useful in our next example.

**Example 2, Epicycloids/Hypocycloids**
1. Reset the applet using the Reset Applet button. Then clear the 3D Plot using the **Clear All Graphs** button. Select **Add a Space Curve** from the Graph menu.
2. Zoom out to -8 to 8 in both *x*- and *y*-directions using the Zoom-out button.
3. Now enter the following parametric function: $x = a\cos(t) - \cos(at)$, $y = a\sin(t) - \sin(at)$, $z = 0$.
4. Enter a range of 0 to 2pi for *t*, and select **Restrict view to 2D** just above the **Graph** button on the **Add a Space Curve** dialog. If this has not already graphed the function, click the **Graph** button.
5. The Adjust Parameters dialog should have been displayed. (If not, use the **Parameters** menu option and select Adjust Parameters.) Vary the value of *a* using the scrollbar. Notice that when the value of *a* is a positive integer, you will see an epicycloid with *a* cycles (petals). When the value of *a* is a negative integer, you will see a hypocycloid with *a* cycles (points).
6. Now it would be nicest if we only showed these integer values for the parameter *a*. To do this, select the Range tab on the Adjust Parameters dialog. Set the values of *a* to range from -10 to 10 and then enter 20 steps. This will force *a* to only take on the integer values between -10 and 10, inclusive. Now try the scrollbar again to see the series of epicycloids and hypocycloids displayed one after another.

**Creating an STL file for use with a 3D Printer:**

One of the newest features of CalcPlot3D is its ability to save a surface to an STL file. This STL file can then be used to print a solid 3D surface model using a 3D printer, sometimes called a rapid prototyping machine. For example, here are the saddle, $z = x^2 - y^2$, and the parametric surface I call the Big Knot.



In order for the STL file to work well with the 3D printer, the surface must be given sufficient thickness. To address this issue, CalcPlot3D will automatically give the thickness you specify to a surface before creating the STL file. The applet essentially creates a closed solid to represent the surface.



1.  To give this feature a try, first find a surface that you would like to print to a 3D printer and graph it with CalcPlot3D.
2.  From the **File** menu, select **Save Surfaces in StL File**. A dialog will appear like the one to the right.
3.  Enter the name you wish to give to the solid, and then enter the approximate size you expect to print it at and the desired thickness. These will be used together with the dimensions of the 3D plot window to determine the approximate correct thickness to use with the surface to account for the scaling to be done in the printer application. You may need to experiment a little with these values to get the thickness just right.
4.  If the surface you wish to print is already a closed 3D surface like a sphere or a torus, you can specify that no thickness be given to the surface before printing, or you can still use a thickness, depending on how you desire the printer to fill the void inside the closed surface. In this case, if you choose to give the closed surface a thickness, you will need to choose between using the **Opposite Normals** direction (generally inside the closed solid), the **Normal Direction** (generally outside the closed surface), or the **Half-and-Half** option, which goes both directions half the thickness and creates two new shells with the same basic shape.
5.  The **Half-and-Half** option is always best for surfaces that flip orientation in the middle of a surface like a Möbius Strip or a Klein Bottle.
6.  Use the **View Solid** button to see what the surface/solid will look like before saving the STL file.
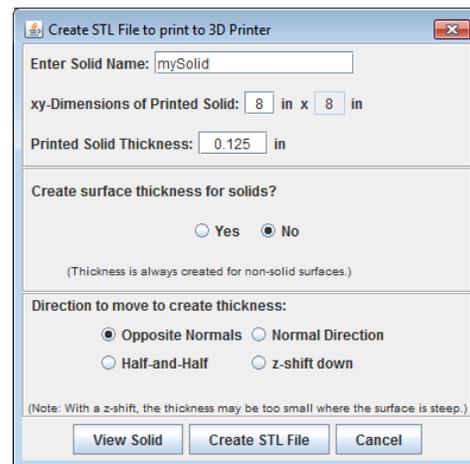
7.  For regular surfaces given by functions of two variables like the saddle above, I prefer the *z*-shift down option since it preserves the exact shape of the surface on the bottom as well as the top of the printed surface. But you need to compensate for the fact that the thickness will vary over the surface and be least thick where the surface is steepest in the *z*-direction.

8.  Once you are ready, click on the Create STL File button. You will be asked for the file name and location and then the applet will create the STL file using the ASCII STL format (the binary STL format may become the default eventually). A small progress bar dialog may appear to show how it is going. Unless you use a very large number of polygons to make your surface smoother, you may not even see this progress bar, since it will be fairly fast.

9.  Once you have created the file, it may be helpful to download an STL viewer from either Marcam Engineering (free viewer is called VisCAM RP) or Materialise (free viewer called MiniMagics 2.0). I prefer the first right now. This allows you to check the surface for thickness and for other possible problems before you print it.

10. Now you should be ready to print to the 3D printer using the 3D printer's software. There is still some room to play with the surface in this software, scaling it appropriately and choosing how you want to fill the surface and how much support material to use. Ask your local expert to help you with this step. By the way, it can take a long time to print these surfaces. My 6 in. x 6 in. saddle surface above took 13 hours to print and the 4 in. x 4 in. knot took almost 9 hours.

Have fun and explore! Please let me know what uses you find for this applet and what features or surface (curve) examples you would like to see added.