April 2018 Puzzle

To reward his son for good behavior, Isaac gave his son quarters for N consecutive days. On the first day, he gave him 1 quarter and then 1/7th of the quarters that remained. On the second day, he gave him 2 quarters and then 1/7th of the quarters that remained. On the third day, he gave him 3 quarters and then 1/7th of the quarters that remained. This continued until the N^{th} day, when he gave his son N quarters and ran out of quarters. How many total quarters did Isaac give his son and over how many days did this occur?

April 2018 Solution

Answer: Isaac gave his son a total of 36 quarters over 6 consecutive days.

To first demonstrate that the answer above is possible, we show how the money would have been given out over the 6 day period.

Day	Number of Quarters Given	Number of Quarters Remaining to be Given at the End of the Day
1	$1+\frac{1}{7}(36-1)=6$	$36-6=30$
2	$2+\frac{1}{7}(30-2)=6$	$30-6=24$
3	$3+\frac{1}{7}(24-3)=6$	$24-6=18$
4	$4+\frac{1}{7}(18-4)=6$	$18-6=12$
5	$5+\frac{1}{7}(12-5)=6$	$12-6=6$
6	6	$6-6=0$

While the table above shows that giving out 36 quarters over 6 consecutive days is a possible answer, it doesn't prove that no other answer is possible. We will now show that the above answer is the only possible answer. To do so, we first introduce the following notation:

$N =$ the number of consecutive days that the quarters are given out for

$T_0 =$ the total number of quarters to be given to the son

$T_i =$ the number of quarters remaining to be given at the end of day i

We first note that on day i, Isaac will give his son i quarters plus 1/7th of the number of quarters that remain. The number of coins remaining at the end of the previous day will be T_{i-1} and therefore, on day i, Isaac will give his son $i+\frac{1}{7}(T_{i-1}-i)$ quarters. As a result we have

$$T_i = T_{i-1} - \left[i + \frac{1}{7}(T_{i-1}-i)\right] = T_{i-1} - i - \frac{1}{7}(T_{i-1}-i) = \frac{6}{7}(T_{i-1}-i)$$

The result above applies for $i = 1, 2, \cdots, N$. Solving $T_i = \frac{6}{7}(T_{i-1} - i)$ for T_{i-1}, we have:

$$T_{i-1} = \frac{7 \cdot T_i}{6} + i \quad \text{for } i = 1, 2, \cdots, N$$

We know that $T_N = 0$ and therefore,

$$T_{N-1} = \frac{7 \cdot T_N}{6} + N = N$$

$$T_{N-2} = \frac{7 \cdot T_{N-1}}{6} + N - 1 = N\left(\frac{7}{6}\right) + N - 1$$

$$T_{N-3} = \frac{7 \cdot T_{N-2}}{6} + N - 2 = \frac{7}{6}\left[\frac{7N}{6} + N - 1\right] + N - 2 = N\left(\frac{7}{6}\right)^2 + (N-1)\left(\frac{7}{6}\right) + (N-2)$$

$$\vdots$$

$$T_0 = T_{N-N} = N\left(\frac{7}{6}\right)^{N-1} + (N-1)\left(\frac{7}{6}\right)^{N-2} + \cdots + 2\left(\frac{7}{6}\right) + 1$$

We now make use of the following summation formula whose derivation will be provided at the end of this solution for the interested reader.

$$1 + 2r + 3r^2 + \cdots + nr^{n-1} = \frac{[n(r-1) - 1]r^n + 1}{(r-1)^2} \qquad (r \neq 1)$$

In our case, we have $r = \frac{7}{6}$ and $n = N$ and therefore

$$T_0 = \frac{\left[\frac{1}{6}N - 1\right]\left(\frac{7}{6}\right)^N + 1}{\left(\frac{1}{6}\right)^2} = [6N - 36]\left(\frac{7}{6}\right)^N + 36 = \frac{N-6}{6^{N-1}} \cdot 7^N + 36$$

We know that T_0 is a positive integer and that $N \geq 3$, since we were told that Isaac gave his son quarters on the third day. When $N \geq 3$, the expression $\frac{N-6}{6^{N-1}}$ will be a proper fraction that when in lowest terms will have a denominator containing only factors 2 and 3. Since 7^N will never contain factors of 2 or 3, the expression $\frac{N-6}{6^{N-1}} \cdot 7^N$ will not be an integer except when $N = 6$ where it has a value of 0. We conclude that the only value of $N \geq 3$ which results in T_0 being a positive integer is $N = 6$ which results in $T_0 = 36$. This result agrees with our answer that Isaac gave his son 36 quarters over 6 consecutive days.

Derivation of the Summation Formula Used Above.

The summation formula used in our solution can be derived in multiple ways. We provide a derivation that avoids the use of calculus. Throughout we assume $r \neq 1$.

We start by considering what is known as geometric sum which takes the form below.

$$S = 1 + r + r^2 + \cdots + r^{n-2} + r^{n-1}$$

Multiplying both sides by r results in

$$rS = r + r^2 + r^3 + \cdots + r^{n-1} + r^n$$

Subtracting the first equation from the second gives us

$$rS - S = r + r^2 + r^3 + \cdots + r^{n-1} + r^n - (1 + r + r^2 + \cdots + r^{n-2} + r^{n-1})$$

Simplifying we obtain $S(r-1) = r^n - 1$ and therefore, $S = 1 + r + r^2 + \cdots + r^{n-2} + r^{n-1} = \frac{r^n - 1}{r-1}$.

Now consider the sum

$$T = 1 + 2r + 3r^2 + \cdots + (n-1)r^{n-2} + nr^{n-1}$$

Multiplying both sides by r results in

$$\begin{aligned} rT &= r + 2r^2 + 3r^3 + \cdots + (n-1)r^{n-1} + nr^n \\ &= (2-1)r + (3-1)r^2 + (4-1)r^3 + \cdots + (n-1)r^{n-1} + nr^n \\ &= 2r + 3r^2 + 4r^3 + \cdots + nr^{n-1} - r - r^2 - r^3 - \cdots - r^{n-1} + nr^n \end{aligned}$$

Subtracting the equation for T from the equation for rT and canceling common terms gives us

$$rT - T = nr^n - (1 + r + r^2 + r^3 + \cdots + r^{n-1})$$

The expression in parentheses is the geometric sum that we showed is equal to $\frac{r^n - 1}{r-1}$.

Therefore, the last equation above can be rewritten as $T(r-1) = nr^n - \frac{r^n - 1}{r-1} = \frac{[n(r-1)-1]r^n + 1}{r-1}$.

Solving for T we have

$$T = \frac{[n(r-1)-1]r^n + 1}{(r-1)^2}$$

This establishes the formula $1 + 2r + 3r^2 + \cdots + nr^{n-1} = \frac{[n(r-1)-1]r^n + 1}{(r-1)^2}$