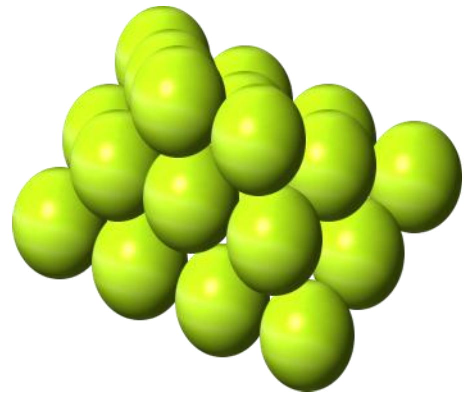
March 2018 Puzzle

A large collection of spheres, each with a radius of 1 inch are stacked as described below:



The bottom row consists of a 1000 × 1001 rectangular array of spheres sitting on a flat table so that each sphere touches its neighbors. On top of this layer lies a 999 × 1000 rectangular array of touching spheres, each which lies in the "dimples" created by the layer below. This continues until one reaches the top where we have a 1 × 2 rectangular array of spheres. The top three layers are shown in the figure to the right.

Determine the total number of spheres that are stacked and the total height of the stack.

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Answer: The stack has a total of 334,334,000 spheres and is $2 + 999\sqrt{2} \approx 1415$ inches high.

The total number of spheres in the stack is given by

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + 1000 \times 1001$$

Before computing the value of the expression above, we first note that the expression clearly indicates that there are 1000 layers in the stack.

While there is a formula for sums of the form $1 \times 2 + 2 \times 3 + \cdots + n \times (n+1)$, we will obtain the total using two more commonly known sum formulas. Every term in the sum takes the form $k(k+1)$. But $k(k+1) = k^2 + k$ and thus we will use the following summation formulas

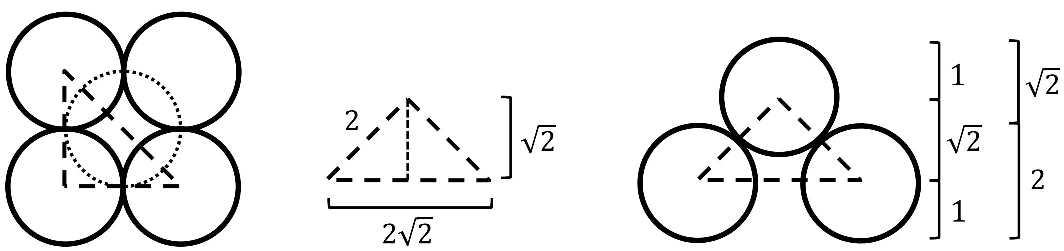
$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \quad \text{and} \quad 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

As a result we have,

$$\begin{aligned} 1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + 1000 \times 1001 &= (1^2 + 1) + (2^2 + 2) + (3^2 + 3) + \cdots + (1000^2 + 1000) \\ &= (1 + 2 + 3 + \cdots + 1000) + (1^2 + 2^2 + 3^2 + \cdots + 1000^2) \\ &= \frac{1000 \times 1001}{2} + \frac{1000 \times 1001 \times 2001}{6} \\ &= 334{,}334{,}000 \end{aligned}$$

We now determine the total height of the stack, recalling that there are 1000 layers in the stack. Since the radius of each sphere is 1 inch, the bottom layer (and the others) is 2 inches high. The spheres in each successive layer lie in the "dimples" created by the four spheres each touches in the layer below it. Therefore, we must determine how much each additional layer adds to the overall height.

The leftmost figure below shows the top view of a single sphere lying in the dimple of the four spheres in the layer below it. The center of this single sphere lies in the same plane as the centers of each pair of diagonally opposing spheres below it. Using the fact that each sphere has radius 1 inch, we see that the centers of the diagonally opposing spheres are $2\sqrt{2}$ inches apart. The middle figure below shows that the center of the single sphere in the top layer is $\sqrt{2}$ inches above the centers of the spheres in the layer below. The rightmost figure then shows that each additional layer contributes an additional $\sqrt{2}$ inches to the overall height of the stack.



Since the radius of each sphere is 1 inch, the bottom layer is 2 inches tall. Each of the remaining 999 layers adds an additional $\sqrt{2}$ inches to the overall height. Therefore, the height of the stack is

$$2 + 999\sqrt{2} \approx 1415 \text{ inches}$$