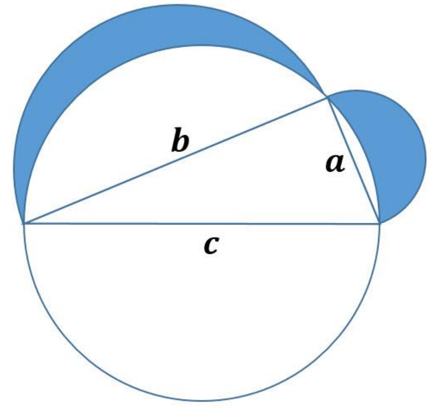
April 2017 Puzzle

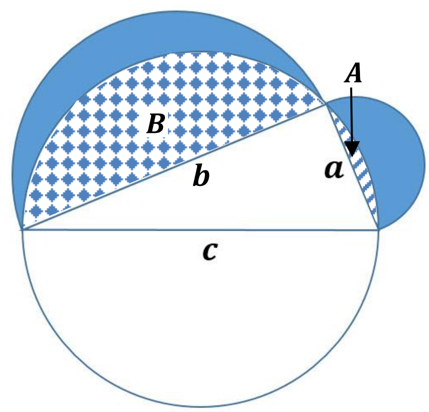
Given a right triangle with sides of length *a, b,* and *c*, first construct the circle whose diameter is the hypotenuse of length *c*. It is known from *Euclidean Geometry* that all three vertices of this triangle will lie on the circle. Next form the (outward facing) semicircles whose diameters are the legs of the triangle. The result is shown in the figure to the right. The shaded regions that lie outside of the big circle and inside the smaller circles are called *lunes*. Let *L* be the total area of the two lunes and let *T* be the area of the right triangle. Determine the value of *L/T*.



April 2017 Solution

The answer is: $\boldsymbol{\frac{L}{T} = 1}$

The area of the upper-semicircle whose diameter is c can be broken down as the sum of the areas of the right triangle and the two shaded regions whose areas we will denote A and B, as indicated in the diagram to the right. As a result, we obtain



$$\frac{1}{2}\pi\left(\frac{c}{2}\right)^2 = \frac{1}{2}ab + A + B$$

Therefore, $A + B = \frac{\pi}{8}c^2 - \frac{1}{2}ab$.

The total area of the two lunes can be obtained by subtracting $A + B$ from the total area of the two semicircles whose diameters are a and b. This gives us

$$L = \frac{1}{2}\pi\left(\frac{a}{2}\right)^2 + \frac{1}{2}\pi\left(\frac{b}{2}\right)^2 - (A + B) = \frac{\pi}{8}(a^2 + b^2) - (A + B)$$

By the *Pythagorean Theorem*, we know that $a^2 + b^2 = c^2$. Using this fact along with the expression obtained for $A + B$ above, we can re-express the total area of the lunes as

$$L = \frac{\pi}{8}c^2 - \left(\frac{\pi}{8}c^2 - \frac{1}{2}ab\right) = \frac{1}{2}ab$$

This shows that the total area of the lunes is equal to the area of the right triangle.

Therefore, $\frac{L}{T} = \frac{\frac{1}{2}ab}{\frac{1}{2}ab} = 1$.