February 2017 Puzzle

The interior of rectangular tank has a square base with sides of length10 feet. Two-hundred cubic feet of water is poured into the tank resulting in a water-level of 2 feet. A solid rectangular block, dense enough to sink to the bottom of the tank, is placed in the water three times. Each time the block was placed with a different face touching the bottom and the resulting water-levels were recorded to be 4 feet, 5 feet, and 6 feet. Determine the dimensions of the block.

February 2017 Solution

The answer is: $\mathbf{3\sqrt{5}\ ft \times \frac{10\sqrt{5}}{3}\ ft \times 4\sqrt{5}\ ft}$ or approximately $\mathbf{6.71\ ft \times 7.45\ ft \times 8.94\ ft}$

We are given that the 200 ft^3 of water results in a water-level of 2 feet when the block is not in the tank. Therefore, each foot of depth contributes 100 ft^3 of volume.

Let the sides of the block have lengths x, y, and z. Because the water-levels were different when each face touched the bottom, we know that the block couldn't have been fully submerged in more than one of those placements.

Let us first consider the possibility that the block wasn't fully submerged during any of the placements.

When the face with dimensions $x \times y$ was on the bottom, the water-level rose 2 ft to a level of 4 ft and therefore, the volume of the portion of the block in the water is given by

$$4xy = 200 \quad \rightarrow \quad xy = 50$$

When the face with dimensions $x \times z$ was on the bottom, the water-level rose 3 ft to a level of 5 ft and therefore, the volume of the portion of the block in the water is given by

$$5xz = 300 \quad \rightarrow \quad xz = 60$$

When the face with dimensions $y \times z$ was on the bottom, the water-level rose 4 ft to a level of 6 ft and therefore, the volume of the portion of the block in the water is given by

$$6yz = 400 \quad \rightarrow \quad yz = \frac{200}{3}$$

Multiplying the three equations obtained above together we obtain

$$x^2y^2z^2 = 200{,}000 \quad \rightarrow \quad xyz = \sqrt{200{,}000} \quad \rightarrow \quad xyz = 200\sqrt{5}$$

Therefore, $x = \frac{200\sqrt{5}}{yz} = \frac{200\sqrt{5}}{200/3} = 3\sqrt{5}$, $y = \frac{200\sqrt{5}}{xz} = \frac{200\sqrt{5}}{60} = \frac{10\sqrt{5}}{3}$, and $z = \frac{200\sqrt{5}}{xy} = \frac{200\sqrt{5}}{50} = 4\sqrt{5}$ which are the values given in the solution to the puzzle.

The above calculations assumed that the block was never fully submerged during any of the placements. We will now show that this must have been the case.

Suppose that the block was fully submerged only when the face with dimensions $x \times y$ was on the bottom. In this case the equations $xz = 60$ and $yz = 200/3$ would still hold true. However, the resulting water-level of 4 ft tells us that the total volume of the water and the block is given by

$$200 + xyz = 400 \quad \rightarrow \quad xyz = 200$$

It would then follow that

$$x = \frac{200}{yz} = \frac{200}{200/3} = 3, \quad y = \frac{200}{xz} = \frac{200}{60} = \frac{10}{3}, \quad \text{and} \quad z = \frac{200}{xy} = \frac{200}{3(10/3)} = 20$$

The result of $z = 20$ is not compatible with our assumption of the block being fully submerged when the face with dimensions $x \times y$ was on the bottom—that would require $z < 4$.

A similar result arises when one supposes that the block was fully submerged with either of the other two faces on the bottom and therefore it follows that the block couldn't have been fully submerged during any of the placements.