April 2016 Puzzle

Tim is a mathematician with two rather intelligent children, Matthew and Kristen. One day, he offers them the following challenge. He tells them that he has a triangle whose side lengths are all integers. He tells only Matthew that the perimeter is 11 and tells only Kristen the area. Each is asked to independently determine the three side lengths of the triangle and each is intelligent enough to do so if they are given sufficient information. After a few minutes Tim asks each of the children if they have come up with the answer and each indicates that it is impossible to determine with certainty. At that point, Matthew thinks for a moment and with no further information says that he now knows the answer with certainty. At that point, Kristen then thinks for a moment and indicates that she now knows with certainty. What were the lengths of the three sides of the triangle?

April 2016 Solution

The answer is… 3, 4, and 4.

As we are searching for a triangle whose side lengths are all integers, we will refer to such triangles as *integral* triangles. To arrive at our result, we first note the sum of the lengths of any two sides of a triangle must always be greater than the length of the remaining side (this fact is called the *Triangle Inequality*). Using this fact we see that the only integral triangles with perimeter 11 are those with the following side lengths:

$$(1,5,5), \quad (2,4,5), \quad (3,3,5), \quad (3,4,4)$$

Matthew is able to determine that the triangle must correspond to one of the four cases above, but initially he cannot deduce which case it is. Knowing that the triangle corresponds to one of the cases above, he is able to determine the possibilities for the area that Kristen was given. To calculate these areas, we will use the variation of *Heron's Formula* given below.

Given a triangle with sides of length a, b, and c and perimeter, $P = a + b + c$, then the area of the triangle is

$$A = \frac{1}{4}\sqrt{P(P-2a)(P-2b)(P-2c)}$$

This gives us the following four possible areas that Kristen was given. In each case, we will identify the radicand which will later be used to deduce exactly which triangle the father chose.

- $(1,5,5) \rightarrow \quad A = \frac{1}{4}\sqrt{11(9)(1)(1)} = \frac{1}{4}\sqrt{99} \quad \rightarrow \quad \text{radicand} = 99$
- $(2,4,5) \rightarrow \quad A = \frac{1}{4}\sqrt{11(7)(3)(1)} = \frac{1}{4}\sqrt{231} \quad \rightarrow \quad \text{radicand} = 231$
- $(3,3,5) \rightarrow \quad A = \frac{1}{4}\sqrt{11(5)(5)(1)} = \frac{1}{4}\sqrt{275} \quad \rightarrow \quad \text{radicand} = 275$
- $(3,4,4) \rightarrow \quad A = \frac{1}{4}\sqrt{11(5)(3)(3)} = \frac{1}{4}\sqrt{495} \quad \rightarrow \quad \text{radicand} = 495$

Because the radicands associated the areas above can be factored as $P(P-2a)(P-2b)(P-2c)$, we can now consider all positive integer factorizations of each of the radicands we obtained. Before doing so, let us observe the last three factors sum to the first factor as shown below:

$$(P-2a)+(P-2b)+(P-2c)=3P-2(a+b+c)=3P-2P=P$$

Let's first consider the radicand, 99. The only factorizations of 99 into four positive integer factors are as follows (the order of the last three factors is irrelevant, but the first must be the largest):

$$99(1)(1)(1); \quad 33(3)(1)(1); \quad 11(9)(1)(1); \quad 11(3)(3)(1)$$

The only case above where the last three factors sum to the first is the third case, which indicates that there is only one integral triangle whose area is $\frac{1}{4}\sqrt{99}$. If Kristen were given this area, then she would have been able to deduce that the triangle must have sides of length 1, 5, and 5.

Similarly, a radicand of 231 only admits factorizations:

$$231(1)(1)(1); \quad 77(3)(1)(1); \quad 33(7)(1)(1); \quad 21(11)(1)(1); \quad 11(7)(3)(1)$$

Once again, only one of these factorizations (the last one) satisfies that the last three factors sum to the first and therefore if Kristen were given an area of $\frac{1}{4}\sqrt{231}$ she would know that the triangle would have sides of length 2, 4, and 5.

The radicand 275 admits factorizations:

$$275(1)(1)(1); \quad 55(5)(1)(1); \quad 25(11)(1)(1); \; 11(5)(5)(1)$$

As before, the only factorization that could have come from our area formula is the last one. So if Kristen were given an area of $\frac{1}{4}\sqrt{275}$, she would be able to deduce that it would have come from a triangle with side lengths 3, 3, and 5.

This leaves us with the final case where the radicand is 495. This radicand admits two factorizations where the sum of the last three factors equals the first (as well as other factorizations that do not satisfy that condition). These two "valid" factorizations are:

$$15(11)(3)(1) \quad \text{and} \quad 11(5)(3)(3)$$

Thus, if Kristen were given an area of $\frac{1}{4}\sqrt{495}$, she would be unable to deduce what the lengths of the triangle are. The first factorization corresponds to a triangle with sides of length 2, 6, and 7 while the second factorization corresponds to a triangle with side lengths 3, 4, and 4.

Once Matthew finds out that Kristen cannot deduce the answer, he knows that she must be given a triangle with area $\frac{1}{4}\sqrt{495}$ and it is only the triangle having sides of length 3, 4, and 4 that has the perimeter 11 that he was given. Thus Matthew is able to determine that the triangle his father had chosen had sides of length 3, 4, and 4 as claimed.

While it is not necessary to indicate how Kristen was able to determine the answer once Matthew arrived at his answer, we will do so for completeness.

We know that Kristen was given the area, $\frac{1}{4}\sqrt{495}$, which based on the factorizations provided above, only leads to possible perimeters of 11 or 15. If Matthew was given a perimeter of 15, then he would have concluded that the triangle would have to have sides with lengths corresponding to one of the following cases:

$$(1,7,7), \quad (2,6,7), \quad (3,6,6), \quad (4,5,6), \quad (4,4,7), \quad (5,5,5)$$

This time we will just look at the cases corresponding to $(2,6,7)$ and $(4,4,7)$ which yield the following area calculations:

- $(2,6,7) \rightarrow \quad A = \frac{1}{4}\sqrt{15(11)(3)(1)} = \frac{1}{4}\sqrt{495} \quad \rightarrow \quad \text{radicand} = 495$
- $(4,5,6) \rightarrow \quad A = \frac{1}{4}\sqrt{15(7)(5)(3)} = \frac{1}{4}\sqrt{1575} \quad \rightarrow \quad \text{radicand} = 1575$

The radicand 495 also admits a factorization as $11(5)(3)(3)$ which would correspond to a triangle with sides of length 1, 4, and 4. The radicand 1575 also admits a factorization as $21(15)(5)(1)$ which would correspond to a triangle with sides of length 3, 8, and 10. What this tells us is that if Matthew were given the perimeter 15, he would not have been able to determine what the side lengths were based on Kristen not knowing, since Kristen not knowing could have been a result of either of the two cases above.

Thus, once Kristen found out that Matthew was able to determine the perimeter once she indicated she didn't know, she was able determine the perimeter must be 11. From there, she was able to conclude the side lengths based on the fact that only one integral triangle has a perimeter of 11 and area of $\frac{1}{4}\sqrt{495}$.

rev. 10/2018